

# Free Field Representation of Quantum Affine Algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and Form Factors in Higher Spin XXZ Model

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## ABSTRACT

We consider the spin  $k/2$  XXZ model in the antiferromagnetic regime using the free field realization of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  of level  $k$ . We give a free field realization of the type II  $q$ -vertex operator, which describes creation and annihilation of physical particles in the model. By taking a trace of the type I and the type II  $q$ -vertex operators over the irreducible highest weight representation of  $U_q(\widehat{\mathfrak{sl}}_2)$ , we also derive an integral formula for form factors in this model. Investigating the structure of poles, we obtain a residue formula for form factors, which is a lattice analog of the higher spin extension of the Smirnov's formula in the massive integrable quantum field theory. This result as well as the quantum deformation of the Knizhnik-Zamolodchikov equation for form factors shows a deep connection in the mathematical structure of the integrable lattice models and the massive integrable quantum field theory.

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# 1 Introduction

Recently, in the attempts to clarify the mathematical structure of the massive integrable quantum field theory (MIQFT), quantum deformation of the affine Lie algebra is recognized as a key symmetry to the integrability of the theory [1, 2]. Especially, Smirnov has argued that his equation for form factors in  $SU(2)$  invariant Thirring model is nothing but a quantum deformation of the Knizhnik-Zamolodchikov (KZ) equation and that the Yangian double, a kind of quantum deformation of the affine Lie algebra  $\widehat{\mathfrak{sl}}_2$  with zero central charge, is the key symmetry to the equation. ( See also the third paper in Ref.[2])

However, in the Smirnov's work, the solutions of the quantum deformed KZ equation, i.e form factors, are not well understood in the language of the representation theory of the Yangian.

On the other hand, Frenkel and Reshetikhin[3] derived a quantum deformation of the KZ equation based on the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  and related it to integrable lattice models. Further, using the representation of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$ , people in Kyoto school reformulated the XXZ model in the antiferromagnetic regime[4, 5] and extended it to the higher spin cases[6]. There, in contrast to the Smirnov's works, the form factors are well defined in terms of vertex operators and highest weight modules.

An important observation made in [4, 6] is that the quantum KZ equation for the form factors in the XXZ model coincides with Smirnov's deformed KZ equation in a certain scaling limit. This suggests a deep connection in the mathematical structures between the XXZ model and the  $SU(2)$  invariant Thirring model, especially a possibility of understanding the Smirnov's results in terms of the representation theory of quantum affine algebra.

The main purpose of this paper is to inquire further into this connection. More precisely, we show that the form factor defined by Idzumi et al.[6] based on the representation theory of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  satisfies a lattice analog of the axioms for form factors in MIQFT invented by Smirnov[7].

The Smirnov's axioms are summarized as follows.

Axiom 1 Form factors  $f(\beta_1, \dots, \beta_n)_{\epsilon_1, \dots, \epsilon_n}$  have the S-matrix symmetry

$$\begin{aligned} f(\beta_1, \dots, \beta_i \beta_{i+1}, \dots, \beta_n)_{\epsilon_1, \dots, \epsilon_i \epsilon_{i+1}, \dots, \epsilon_n} S_{\epsilon_i \epsilon_{i+1}}^{\epsilon'_i \epsilon'_{i+1}}(\beta_i - \beta_{i+1}) \\ = f(\beta_1, \dots, \beta_{i+1} \beta_i, \dots, \beta_n)_{\epsilon_1, \dots, \epsilon'_{i+1} \epsilon'_i, \dots, \epsilon_n} \end{aligned} \quad (1.1)$$

Axiom 2 Form factors  $f(\beta_1, \dots, \beta_n)_{\epsilon_1, \dots, \epsilon_n}$  satisfy the difference equation

$$f(\beta_1, \dots, \beta_n + 2\pi i)_{\epsilon_1, \dots, \epsilon_n} = f(\beta_n, \beta_1, \dots, \beta_{n-1})_{\epsilon_n, \epsilon_1, \dots, \epsilon_{n-1}} \quad (1.2)$$

Axiom 3 Form factors  $f(\beta_1, \dots, \beta_n)_{\epsilon_1, \dots, \epsilon_n}$  have simple poles at  $\beta_i = \beta_j + \pi i$ ,  $i > j$ . The corresponding residues at  $\beta_n = \beta_{n-1} + \pi i$  are given by

$$\begin{aligned}
& 2\pi i \operatorname{res} f(\beta_1, \dots, \beta_n)_{\epsilon_1, \dots, \epsilon_n} \\
&= f(\beta_1, \dots, \beta_{n-2})_{\epsilon'_1, \dots, \epsilon'_{n-2}} \delta_{\epsilon_n, -\epsilon_{n-1}} \\
&\times \left( \delta_{\epsilon'_1}^{\epsilon'_1} \dots \delta_{\epsilon'_{n-2}}^{\epsilon'_{n-2}} - S_{\tau_1 \epsilon'_1}^{\epsilon'_{n-1} \epsilon'_1} (\beta_{n-1} - \beta_1) \dots S_{\epsilon_{n-1} \epsilon'_{n-2}}^{\tau_{n-3} \epsilon'_{n-2}} (\beta_{n-1} - \beta_{n-2}) \right). \tag{1.3}
\end{aligned}$$

These axioms define the matrix Riemann-Hilbert problem, which provides complete informations to derive integral formulae for form factors of local operators in various MIQFTs[7]. Especially, one should note that the deformed KZ equation for form factors is derived from the first and the second axioms.

In Refs.[4, 5, 6], the correlation functions and the form factors in the XXZ model were reconstructed as a trace of the product of some  $q$ -vertex operators over an irreducible highest weight representation (IHWR) of  $U_q(\widehat{\mathfrak{sl}}_2)$ . As a result, the lattice analogs of the first and the second axioms are understood as an  $R$ -matrix symmetry of the  $q$ -vertex operators and the cyclic property of trace. We here consider the third axiom in the same spirit. Namely, we verify the lattice analog of the third axiom in terms of the  $q$ -vertex operators. In the case of the spin 1/2 XXZ model, the same subject was discussed by Pakuliak[8]. However, his analysis is based on the result restricted to the spin 1/2 case. In this paper, we consider more general situations containing the higher spin extensions.

We also discuss an explicit calculation of form factors in the spin  $k/2$  XXZ model. Even though this is not necessary to verify the third axiom, this subject is important in the physical application. We explicitly evaluate a trace of the  $q$ -vertex operators over IHWR of  $U_q(\widehat{\mathfrak{sl}}_2)$  and obtain an integral formula for form factors.

For these purposes, we make a full use of the free field realization of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  of level  $k(\geq 1)$ [9, 10, 11, 12]. This realization is a higher level extension of the level one realization obtained in Ref.[13, 5]. The characteristic feature of our realization is the existence of screening operators and their Jackson integrals ( screening charges ) in the expression of the  $q$ -vertex operators. For the purpose of physical application, it is required to determine the cycles for the Jackson integrals. We discuss the determination of the cycles, which leads to the  $R$ -matrix symmetry of the  $q$ -vertex operators.

Furthermore, as in the free field representation of the affine Lie algebras[14, 15], the Fock space representation ( quantum deformation of the Wakimoto module[16] ) of  $U_q(\widehat{\mathfrak{sl}}_2)$  is, in general, reducible. This makes no sense in the simple minded evaluation of traces over the  $q$ -Wakimoto modules. In Ref.[12], extending the Felder's BRST analysis of the affine Lie algebra  $\widehat{\mathfrak{sl}}_2$  to the case  $U_q(\widehat{\mathfrak{sl}}_2)$ , we discussed a resolution of IHWR of  $U_q(\widehat{\mathfrak{sl}}_2)$  and gave a formula, which allows us to evaluate a trace over the IHWR in terms of free fields. Our trace calculation is totally based on this result.

This paper is organized as follows. In the next section, we briefly review the free field realization of  $U_q(\widehat{\mathfrak{sl}}_2)$  and the resolution of IHWR based on the BRST-Felder cohomology analysis. The prescriptions of evaluating trace over IHWR are also presented.

In Sec.3, we derive a free field realization of the type II  $q$ -vertex operator of  $U_q(\widehat{\mathfrak{sl}}_2)$  of level  $k$ , which remains to be done in Refs.[9, 10, 11, 12]. This vertex plays a role of an annihilation operator (it's anti-pode dual plays a creation operator) of physical particle in the spin  $k/2$  XXZ model. We determine suitable cycles for the Jackson integral associated with the screening charges by demanding commutativity with all the currents realizing  $U_q(\widehat{\mathfrak{sl}}_2)$ . Calculating two point functions of the type II  $q$ -vertex operators, we show that these cycles imply the  $R$ -matrix symmetry of the type II  $q$ -vertex operators.

In Sec.4, we make an explicit evaluation of the trace of the  $q$ -vertex operators over IHWR for  $U_q(\widehat{\mathfrak{sl}}_2)$ . We give a general integral formula for form factors.

In Sec.5, we investigate a pole structure of form factors. It is shown that only a simple pole appears, when two spectral parameters close each other. Evaluating residues, we obtain a higher spin extension of a lattice analog of the Smirnov's residue formula for MIQFT ( the third axiom (1.3)).

The final section is devoted to a discussion of physical implication of the results.

## 2 Free Field Realization of $U_q(\widehat{\mathfrak{sl}}_2)$

We here briefly review the free field realization of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  and its Fock space representation[9, 10, 11, 12].

### 2.1 Drinfeld realization of $U'_q(\widehat{\mathfrak{sl}}_2)$

The quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  consists of the algebra  $U'_q(\widehat{\mathfrak{sl}}_2)$  and the grading operator  $L_0$ . Let us first consider  $U'_q(\widehat{\mathfrak{sl}}_2)$ . We follow the Drinfeld realization of  $U'_q(\widehat{\mathfrak{sl}}_2)$ [17], which is

generated by the letters  $\{J_n^\pm | n \in \mathbf{Z}\}$ ,  $\{J_n^3 | n \in \mathbf{Z}_{\neq 0}\}$ ,  $\gamma^{\pm 1/2}$  and  $K$  under the relations

$$\begin{aligned}
& \gamma^{\pm 1/2} \in \text{the center of the algebra,} \\
& [J_n^3, J_m^3] = \delta_{n+m,0} \frac{1}{n} [2n] \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \\
& [J_n^3, K] = 0, \\
& K J_n^\pm K^{-1} = q^{\pm 2} J_n^\pm, \\
& [J_n^3, J_m^\pm] = \pm \frac{1}{n} [2n] \gamma^{\mp |n|/2} J_{n+m}^\pm, \\
& J_{n+1}^\pm J_m^\pm - q^{\pm 2} J_m^\pm J_{n+1}^\pm = q^{\pm 2} J_n^\pm J_{m+1}^\pm - J_{m+1}^\pm J_n^\pm, \\
& [J_n^+, J_m^-] = \frac{1}{q - q^{-1}} (\gamma^{(n-m)/2} \psi_{n+m} - \gamma^{(m-n)/2} \varphi_{n+m}).
\end{aligned} \tag{2.1}$$

Here  $\{\psi_r, \varphi_s | r, s \in \mathbf{Z}\}$  are related to  $\{J_l^3 | l \in \mathbf{Z}_{\neq 0}\}$  as follows.

$$\begin{aligned}
\sum_{n \in \mathbf{Z}} \psi_n z^{-n} &= K \exp \left\{ (q - q^{-1}) \sum_{k=1}^{\infty} J_k^3 z^{-k} \right\}, \\
\sum_{n \in \mathbf{Z}} \varphi_n z^{-n} &= K^{-1} \exp \left\{ -(q - q^{-1}) \sum_{k=1}^{\infty} J_{-k}^3 z^k \right\}.
\end{aligned} \tag{2.2}$$

In (2.1), we used the notation

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

The standard Chevalley generators  $\{e_i, f_i, t_i\}$  are given by the following identification

$$t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = J_0^+, \quad f_1 = J_0^-, \quad e_0 t_1 = J_1^-, \quad t_1^{-1} f_0 = J_{-1}^+. \tag{2.3}$$

Let us introduce three free bosonic fields  $\Phi, \phi$  and  $\chi$  carrying parameters  $L, M, N \in \mathbf{Z}_{>0}$ ,  $\alpha \in \mathbf{R}$  defined by[9]

$$X(L; M, N | z; \alpha) = - \sum_{n \neq 0} \frac{[Ln] a_{X,n}}{[Mn] [Nn]} z^{-n} q^{|n|\alpha} + \frac{L \tilde{a}_{X,0}}{MN} \log z + \frac{L Q_X}{MN}, \tag{2.4}$$

for  $X = \Phi, \phi$  or  $\chi$ . We also often use the notation

$$\begin{aligned}
X(N | z; \alpha) &= X(L; L, N | z; \alpha) \\
&= - \sum_{n \neq 0} \frac{a_{X,n}}{[Nn]} z^{-n} q^{|n|\alpha} + \frac{\tilde{a}_{X,0}}{N} \log z + \frac{Q_X}{N}.
\end{aligned} \tag{2.5}$$

The fields are quantized by the commutation relations

$$\begin{aligned}
[a_{\Phi,n}, a_{\Phi,m}] &= \delta_{n+m,0} \frac{[2n][2m]}{n}, & [\tilde{a}_{\Phi,0}, Q_{\Phi}] &= 2(k+2), \\
[a_{\phi,n}, a_{\phi,m}] &= -\delta_{n+m,0} \frac{[2n][2m]}{n}, & [\tilde{a}_{\phi,0}, Q_{\phi}] &= -4, \\
[a_{\chi,n}, a_{\chi,m}] &= \delta_{n+m,0} \frac{[2n][2m]}{n}, & [\tilde{a}_{\chi,0}, Q_{\chi}] &= 4,
\end{aligned} \tag{2.6}$$

where

$$\tilde{a}_{X,0} = \frac{q - q^{-1}}{2 \log q} a_{X,0},$$

for  $X = \Phi, \phi$  and  $\chi$ , and others commute.

By means of the fields  $\Phi, \phi$  and  $\chi$ , we realize the generators  $J^3(z), J^{\pm}(z), K$  and  $\gamma$  in  $U'_q(\widehat{\mathfrak{sl}}_2)$  as follows.

$$\begin{aligned}
J^3(z) &= {}_{k+2}\partial_z \Phi(k+2|q^{-2}z; -1) + {}_2\partial_z \phi(2|q^{-k-2}z; -\frac{k+2}{2}), \\
J^+(z) &= - : [{}_1\partial_z \exp \{ -\chi(2|q^{-k-2}z; 0) \}] \times \exp \{ -\phi(2|q^{-k-2}z; 1) \} :, \\
J^-(z) &= : [{}_{k+2}\partial_z \exp \{ \Phi(k+2|q^{-2}z; -\frac{k+2}{2}) + \phi(2|q^{-k-2}z; -1) \\
&\quad + \chi(k+1; 2, k+2|q^{-k-2}z; 0) \}] \\
&\quad \times \exp \{ -\Phi(k+2|q^{-2}z; \frac{k+2}{2}) + \chi(1; 2, k+2|q^{-k-2}z; 0) \} :,
\end{aligned} \tag{2.7}$$

$$K = q^{\tilde{a}_{\Phi,0} + \tilde{a}_{\phi,0}}, \quad \gamma = q^k. \tag{2.8}$$

Here we defined the  $q$ -difference operator with parameter  $n \in \mathbf{Z}_{>0}$  by

$${}_n\partial_z f(z) \equiv \frac{f(q^n z) - f(q^{-n} z)}{(q - q^{-1})z}.$$

The mode expansions of the currents  $J^3(z)$  and  $J^{\pm}(z)$  should be set as follows.

$$\sum_{n \in \mathbf{Z}} J_n^3 z^{-n-1} = J^3(z), \tag{2.9}$$

$$\sum_{n \in \mathbf{Z}} J_n^{\pm} z^{-n-1} = J^{\pm}(z). \tag{2.10}$$

Finally, the grading operator  $L_0$  is realized as follows.

$$L_0 = L_0^\Phi + L_0^\phi + L_0^\chi \quad (2.11)$$

$$L_0^\Phi = \frac{\tilde{a}_{\Phi,0}(\tilde{a}_{\Phi,0} + 2)}{4(k+2)} + \sum_{n \geq 1} \frac{n^2}{[2n][ (k+2)n ]} a_{\Phi,-n} a_{\Phi,n}, \quad (2.12)$$

$$L_0^\phi = -\frac{\tilde{a}_{\phi,0}(\tilde{a}_{\phi,0} - 2)}{8} - \sum_{n \geq 1} \frac{n^2}{[2n]^2} a_{\phi,-n} a_{\phi,n}, \quad (2.13)$$

$$L_0^\chi = \frac{\tilde{a}_{\chi,0}(\tilde{a}_{\chi,0} + 2)}{8} + \sum_{n \geq 1} \frac{n^2}{[2n]^2} a_{\chi,-n} a_{\chi,n}. \quad (2.14)$$

## 2.2 Finite Dimensional $U_q(\widehat{\mathfrak{sl}}_2)$ module

Let  $V^{(l)}$  be the  $(l+1)$ -dimensional  $U_q(\widehat{\mathfrak{sl}}_2)$  module with basis  $v_m^{(l)}$ ,  $m = 0, 1, 2, \dots, l$ :

$$\begin{aligned} e_1 v_m^{(l)} &= [m] v_{m-1}^{(l)}, & f_1 v_m^{(l)} &= [l-m] v_{m+1}^{(l)}, & t_1 v_m^{(l)} &= q^{l-2m} v_m^{(l)}, \\ e_0 &= f_1, & f_0 &= e_1, & t_0 &= t_1^{-1} \quad \text{on } V^{(l)}, \end{aligned} \quad (2.15)$$

where  $v_m^{(l)} = 0$  if  $m < 0$  or  $m > l$ . In the case  $l = 1$ , we often use the notations  $v_0^{(1)} = v_+$  and  $v_1^{(1)} = v_-$ .

Let  $V_z^{(l)}$  be the affinization of  $V^{(l)}$ . The action of the Drinfeld generators on  $V_z^{(l)}$  is given by [9]

$$\begin{aligned} \gamma^{\pm 1/2} v_m^{(l)} &= v_m^{(l)}, \\ K v_m^{(l)} &= q^{l-2m} v_m^{(l)}, \\ J_n^+ v_m^{(l)} &= z^n q^{n(l-2m+2)} [l-m+1] v_{m-1}^{(l)}, \\ J_n^- v_m^{(l)} &= z^n q^{n(l-2m)} [m+1] v_{m+1}^{(l)}, \\ J_n^3 v_m^{(l)} &= \frac{z^n}{n} \left\{ [nl] - q^{n(l+1-m)} (q^n + q^{-n}) [nm] \right\} v_m^{(l)}. \end{aligned} \quad (2.16)$$

## 2.3 $q$ -Wakimoto module

The quantum deformation of the Wakimoto module is defined as a certain restriction of the Fock module of the three bosons  $\Phi, \phi$  and  $\chi$  [9, 10, 12]. Let us briefly describe this restriction.

Let us begin with the following Fock module.

$$F_{l,s,t} = \left\{ \prod_{n>0} a_{\Phi,-n} \prod_{n'>0} a_{\phi,-n'} \prod_{n''>0} a_{\chi,-n''} |l; s, t > \right\} \quad (2.17)$$

$$|l; s, t > = \exp \left\{ l \frac{Q_\Phi}{2(k+2)} + s \frac{Q_\phi}{2} + t \frac{Q_\chi}{2} \right\} |0 >. \quad (2.18)$$

Here  $|0 >$  is the vacuum state of the Heisenberg algebra (2.6).

Let  $\lambda_l, l = 0, 1, 2, \dots, k$  be the dominant integral weights of level  $k$ ;  $\lambda_l = (k-l)\Lambda_0 + l\Lambda_1$  where  $\Lambda_0$  and  $\Lambda_1$  are the fundamental weights. One can show that the states  $|\lambda_l\rangle \equiv |l; 0, 0\rangle$  give rise to the highest weight states satisfying  $t_1 |\lambda_l\rangle = q^l |\lambda_l\rangle$ ,  $t_0 |\lambda_l\rangle = q^{k-l} |\lambda_l\rangle$ ,  $e_i |\lambda_l\rangle = 0, i = 0, 1$  and  $q^{L_0} |\lambda_l\rangle = q^{\Delta_{\lambda_l}} |\lambda_l\rangle$ , where  $\Delta_{\lambda_l} = l(l+2)/4(k+2)$ .

Let us first fix the picture of the Fock module  $F_{l,s,t}$  [18]. This is carried out by projecting  $F_{l,s,t}$  onto the sector, on which the eigen value of the operator  $\tilde{a}_{\phi,0} + \tilde{a}_{\chi,0}$  is zero. Then we have the picture fixed Fock module  $F_{l,s,s}$ . The highest weight  $U_q(\widehat{\mathfrak{sl}}_2)$  module  $V(\lambda_l)$  of weight  $\lambda_l$  is embeded in the Fock module  $F_l \equiv \oplus_{s \in \mathbf{Z}} F_{l,s,s}$  [9].

Next let us introduce a conjugate fermionic fields  $\eta(z)$  and  $\xi(z)$  by

$$\eta(z) = \exp \left\{ \chi(2|q^{-k-2}z; 0) \right\}, \quad (2.19)$$

$$\xi(z) = \exp \left\{ -\chi(2|q^{-k-2}z; 0) \right\}. \quad (2.20)$$

One can show that the zero mode of the field  $\eta(z)$ ,  $\eta_0 \equiv \oint \frac{dz}{2\pi i} \eta(z)$  commutes with all the generators in  $U_q(\widehat{\mathfrak{sl}}_2)$  [10, 12].

The  $q$ -Wakimoto module  $W_l$  is then defined from  $F_l$  by the projection onto the kernel of the operator  $\eta_0$ .

$$W_l = \oplus_{s \in \mathbf{Z}} \text{Ker } \eta_0(F_{l,s,s}). \quad (2.21)$$

## 2.4 Definition of $q$ -vertex operators

There are two types of the  $q$ -vertex operators [4]. Let  $V(\lambda)$  be a highest weight  $U_q(\widehat{\mathfrak{sl}}_2)$  module of weight  $\lambda$ . The type I and the type II  $q$ -vertex operators ( $q$ -VOs) are the intertwiner Type I:

$$\begin{aligned} \Phi_\lambda^{\mu V^{(l)}}(z) &= z^{\Delta_\mu - \Delta_\lambda} \tilde{\Phi}_\lambda^{\mu V^{(l)}}(z) \\ \tilde{\Phi}_\lambda^{\mu V^{(l)}}(z) &: V(\lambda) \longrightarrow V(\mu) \hat{\otimes} V_z^{(l)}, \end{aligned} \quad (2.22)$$

Type II:

$$\begin{aligned} \Phi_\lambda^{V^{(l)}\mu}(z) &= z^{\Delta_\mu - \Delta_\lambda} \tilde{\Phi}_\lambda^{V^{(l)}\mu}(z) \\ \tilde{\Phi}_\lambda^{V^{(l)}\mu}(z) &: V(\lambda) \longrightarrow V_z^{(l)} \hat{\otimes} V(\mu). \end{aligned} \quad (2.23)$$

The  $q$ -VOs satisfy the intertwining relations

$$\tilde{\Phi}_\lambda^{\mu V^{(l)}}(z) \circ x = \Delta(x) \circ \tilde{\Phi}_\lambda^{\mu V^{(l)}}(z), \quad (2.24)$$

$$\tilde{\Phi}_\lambda^{V^{(l)}\mu}(z) \circ x = \Delta(x) \circ \tilde{\Phi}_\lambda^{V^{(l)}\mu}(z), \quad \forall x \in U_q(\widehat{\mathfrak{sl}}_2), \quad (2.25)$$

where  $\Delta(x)$  denotes a comultiplication:

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + t_i \otimes e_i, & \Delta(f_i) &= f_i \otimes t_i^{-1} + 1 \otimes f_i, \\ \Delta(t_i) &= t_i \otimes t_i. \end{aligned} \quad (2.26)$$



According to Ref.[6], we take the normalization of the type I and the type II  $q$ -VOs as follows<sup>1</sup>.

$$\tilde{\Phi}_\lambda^{\mu V^{(l)}}(z) |\lambda\rangle = |\mu\rangle \otimes v_m^{(l)} + \cdots, \quad (2.27)$$

for  $\lambda = \lambda_n$  and  $\mu = \sigma(\lambda_n) \equiv n\Lambda_0 + (k-n)\Lambda_1$ , and

$$\tilde{\Phi}_\lambda^{V^{(1)}\mu}(z) |\lambda\rangle = v_{\mp}^{(1)} \otimes |\mu\rangle + \cdots, \quad (2.28)$$

for  $\lambda = \lambda_n$  and  $\mu = \lambda_{\pm} \equiv \lambda_{n\pm 1}$ , respectively.

The free field realization of the  $q$ -VOs can be determined by the intertwining relations (2.24) and (2.25) with the in (2.26) as well as those reexpressed by the Drinfeld's generators[19]:

$$\left. \begin{aligned} \Delta(J_k^+) &= J_k^+ \otimes \gamma^k + \gamma^{2k} K \otimes J_k^+ + \sum_{i=0}^{k-1} \gamma^{(k+3i)/2} \psi_{k-i} \otimes \gamma^{k-i} J_i^+ \\ \Delta(J_{-l}^+) &= J_{-l}^+ \otimes \gamma^{-l} + K^{-1} \otimes J_{-l}^+ + \sum_{i=1}^{l-1} \gamma^{(l-i)/2} \varphi_{-l+i} \otimes \gamma^{-l+i} J_{-i}^+ \end{aligned} \right\} \text{mod } N_- \otimes N_+^2, \\ \left. \begin{aligned} \Delta(J_l^-) &= J_l^- \otimes K + \gamma^l \otimes J_l^- + \sum_{i=1}^{l-1} \gamma^{l-i} J_i^- \otimes \gamma^{(i-l)/2} \psi_{l-i} \\ \Delta(J_{-k}^-) &= J_{-k}^- \otimes \gamma^{-2k} K^{-1} + \gamma^{-k} \otimes J_{-k}^- + \sum_{i=0}^{k-1} \gamma^{i-k} J_{-i}^- \otimes \gamma^{-(k+3i)/2} \varphi_{i-k} \end{aligned} \right\} \text{mod } N_-^2 \otimes N_+, \\ \left. \begin{aligned} \Delta(J_l^3) &= J_l^3 \otimes \gamma^{l/2} + \gamma^{3l/2} \otimes J_l^3 \\ \Delta(J_{-l}^3) &= J_{-l}^3 \otimes \gamma^{-3l/2} + \gamma^{-l/2} \otimes J_{-l}^3 \end{aligned} \right\} \text{mod } N_- \otimes N_+ \quad (2.29)$$

for  $k \in \mathbf{Z}_{\geq 0}$  and  $l \in \mathbf{Z}_{>0}$ . Here  $N_{\pm}$  and  $N_{\pm}^2$  are left  $\mathbf{Q}(q)[\gamma^{\pm}, \psi_r, \varphi_s | r, -s \in \mathbf{Z}_{\geq 0}]$ -modules generated by  $\{J_m^{\pm} | m \in \mathbf{Z}\}$  and  $\{J_m^{\pm} J_n^{\pm} | m, n \in \mathbf{Z}\}$  respectively.

According to Refs.[9, 10, 12], the free field realization is given for the type I  $q$ -VO as follows.

$$\tilde{\Phi}_{\lambda_n}^{\lambda_{n'} V^{(l)}}(z) = g_{\lambda_n}^{\lambda_{n'} V^{(l)}}(z) \sum_{m=0}^l \Psi_{l,m}^{(r)}(z) \otimes v_m^{(l)}, \quad (2.30)$$

$$\Psi_{l,m}^{(r)}(z) = \int_0^{q^{-l}z} d_p t_1 \int_0^{q^{-l}z} d_p t_2 \cdots \int_0^{q^{-l}z} d_p t_r \Psi_{l,m}(z) S(t_1) S(t_2) \cdots S(t_r), \quad (2.31)$$

$$\begin{aligned} \Psi_{l,m}(z) &= \frac{[m]}{[l]!} \oint dw_1 \oint dw_2 \cdots \oint dw_{l-m} \\ &\quad \times [\cdots [\Psi_{l,l}(z), J^-(w_1)]_{q^l}, J^-(w_2)]_{q^{l-2}} \cdots J^-(w_{l-m})]_{q^{2(m+1)-l}}, \end{aligned} \quad (2.32)$$

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<sup>1</sup> In the most part of this paper, we consider only the “spin 1/2” type II  $q$ -VO  $\Phi_{\mu}^{V^{(1)}\lambda}(z)$ . This is due to the fact that even in the spin  $k/2$  XXZ model, only the spin 1/2 physical particle appears[6].

$$\Psi_{l,l}(z) = : \exp \left\{ \Phi(l; 2, k+2 | q^k z; \frac{k+2}{2}) \right\} :, \quad (2.33)$$

where  $2r = n+l-n'$ , and  $g_{\lambda_n}^{\lambda_{n'} V^{(l)}}(z)$  being the normalization function. In (2.31), we introduced the screening operator  $S(z)$ , which commutes with  $U_q(\widehat{\mathfrak{sl}}_2)$  and  $\eta_0$  up to a total difference term. It is explicitly given by

$$\begin{aligned} S(z) = & - : \left[ {}_1\partial_z \exp \left\{ -\chi(2 | q^{-k-2}z; 0) \right\} \right] \\ & \times \exp \left\{ -\phi(2 | q^{-k-2}z; -1) - \Phi(k+2 | q^{-2}z; -\frac{k+2}{2}) \right\} : . \end{aligned} \quad (2.34)$$

Here, the symbol  $\int_0^c d_p t$  denotes the Jackson integral defined by

$$\int_0^c d_p t f(t) = c(1-p) \sum_{m=0}^{\infty} f(cp^m) p^m, \quad (2.35)$$

for  $p \in \mathbf{C}$ ,  $|p| < 1$ , and  $c \in \mathbf{C}^\times$ . In our case  $p \equiv q^{2(k+2)}$ . The free field realization of the type II  $q$ -VO is discussed in the next section.

## 2.5 BRST-Felder cohomology

Let  $n \geq 1, n' \geq 0$  be integers, and  $P, P'$  be coprime positive integers satisfying  $\frac{P}{P'} = k+2$ . Then the  $q$ -Wakimoto module  $W_{n,n'} \equiv W_{l_{n,n'}}$  with  $l_{n,n'} = n - n' \frac{P}{P'} - 1$  is reducible[12]. We are especially interested in the case  $1 \leq n \leq P-1$  and  $0 \leq n' \leq P'-1$ . For this case, the resolution of IHWR can be given as a cohomology group in a BRST like complex of the  $q$ -Wakimoto modules[12].

Let us define the BRST charge  $Q_n$ ,  $n \in \mathbf{Z}_{>0}$ , by

$$Q_n = \int_0^{c\infty} d_p t \int_0^{q^2 t\infty} d_p t_2 \cdots \int_0^{q^{2n} t\infty} d_p t_n S(t) S(t_2) \cdots S(t_n), \quad (2.36)$$

where  $c \in \mathbf{C}^\times$  is arbitrary.

The BRST charge  $Q_n$  satisfies the following properties.

- (i)  $Q_n$  commutes with  $U_q(\widehat{\mathfrak{sl}}_2)$  and  $\eta_0$ .
- (ii)  $Q_n Q_{P-n} = Q_{P-n} Q_n = 0$ .
- (iii) The following infinite sequence

$$\cdots \xrightarrow{Q_n} W_{-n+2P,n'} \xrightarrow{Q_{P-n}} W_{n,n'} \xrightarrow{Q_n} W_{-n,n'} \xrightarrow{Q_{P-n}} W_{n-2P,n'} \xrightarrow{Q_n} W_{-n-2P,n'} \xrightarrow{Q_{P-n}} \cdots \quad (2.37)$$

is a complex.

Constructing the singular and cosingular vectors in the  $q$ -Wakimoto module  $W_{n,n'}$ , one can show that the  $U_q(\widehat{\mathfrak{sl}}_2)$  submodule structure generated by the singular and cosingular vectors in the complex has the same structure as the classical case ( $q=1$ )[12] (see also [20, 21]).

This structure gives rise to the following cohomology groups of the complex (2.37)

$$\text{Ker}Q_n^{[s]}/\text{Im}Q_n^{[s-1]} = \begin{cases} 0 & \text{for } s \neq 0 \\ \mathcal{H}_{n,n'} & \text{for } s = 0 \end{cases}, \quad (2.38)$$

where  $Q_n^{[2a]} = Q_n$  and  $Q_n^{[2a-1]} = Q_{P-n}$  with  $a \in \mathbf{Z}$  and the space  $\mathcal{H}_{n,n'}$  is the IHWR for  $U_q(\widehat{\mathfrak{sl}}_2)$  of highest weight  $\lambda_{l_{n,n'}}$ .

As a by-product, we obtain a formula for the trace over the IHWR of  $U_q(\widehat{\mathfrak{sl}}_2)$

$$\text{Tr}_{\mathcal{H}_{n,n'}} \mathcal{O} = \sum_{s \in \mathbf{Z}} (-)^s \text{Tr}_{W_{n,n'}^{[s]}} \mathcal{O}^{[s]}, \quad (2.39)$$

where the graded physical operator  $\mathcal{O}^{[s]}$  is defined recursively by the relations

$$Q_n^{[s]} \mathcal{O}^{[s]} = \mathcal{O}^{[s+1]} Q_n^{[s]}, \quad \mathcal{O}^{[0]} = \mathcal{O}. \quad (2.40)$$

Furthermore, noting the defining process of the  $q$ -Wakimoto modules from the Fock modules, we relate the traces over the  $q$ -Wakimoto modules to those on the Fock modules:

$$\text{Tr}_{W_{n,n'}^{[s]}} \mathcal{O}^{[s]} = \text{Tr}_{F_{n,n'}^{[s]}} \left( \oint \frac{dw_0}{2\pi i w_0} \xi(w_0) \oint \frac{dw}{2\pi i} \eta(w) \mathcal{O}^{[s]} \right) \Big|_{\tilde{a}_{\phi,0} + \tilde{a}_{\chi,0} = 0}, \quad (2.41)$$

where the restriction  $\tilde{a}_{\phi,0} + \tilde{a}_{\chi,0} = 0$  should be understood on its eigenvalues. The RHS can be evaluated in terms of the free fields.

### 3 Type II $q$ -Vertex Operator

In this section, we discuss a free field realization of the type II  $q$ -vertex operator.

#### 3.1 Free field realization

In the same way as the type I case[9], the key to realizing the type II  $q$ -VO is the intertwining relation (2.25) with the comultiplication formulae (2.26) and (2.29).

Let us write the type II  $q$ -VO as follows.

$$\tilde{\Phi}_{\lambda}^{V^{(l)}\mu}(z) = \sum_{m=0}^l v_m^{(l)} \otimes \tilde{\Phi}_{\lambda,m}^{V^{(l)}\mu}(z). \quad (3.1)$$

Then, from (2.25) and (2.26), we have

$$\tilde{\Phi}_{\lambda,m+1}^{V^{(l)}\mu}(z) = \frac{1}{[m+1]} \left\{ \tilde{\Phi}_{\lambda,m}^{V^{(l)}\mu}(z) e_1 - q^{l-2m} e_1 \tilde{\Phi}_{\lambda,m}^{V^{(l)}\mu}(z) \right\}, \quad (3.2)$$

for  $m = 0, 1, 2, \dots, l-1$ .

Furthermore, by using (2.16), (2.25) and (2.29), we obtain the intertwining relations for  $\tilde{\Phi}_{\lambda,0}^{V^{(l)\mu}}(z)$  as follows.

$$\begin{aligned} [J_n^3, \tilde{\Phi}_{\lambda,0}^{V^{(l)\mu}}(z)] &= -z^n \frac{[nl]}{n} \cdot q^{k(n-|n|/2)} \tilde{\Phi}_{\lambda,0}^{V^{(l)\mu}}(z) \quad n \neq 0, \\ [\tilde{\Phi}_{\lambda,0}^{V^{(l)\mu}}(z), J^-(w)] &= 0, \\ K \tilde{\Phi}_{\lambda,0}^{V^{(l)\mu}}(z) K^{-1} &= q^{-l} \tilde{\Phi}_{\lambda,0}^{V^{(l)\mu}}(z). \end{aligned} \quad (3.3)$$

These follow from  $v_0^{(l)} \otimes V(\mu)$  components of the intertwining relation.

We find that the following vertex operator satisfies the whole relations in (3.3)

$$\begin{aligned} \Phi_{l,0}(z) &= : \exp \left\{ \Phi(l; 2, k+2 | q^{k-2}z; -\frac{k+2}{2}) \right. \\ &\quad \left. + \phi(l; 2, 1 | q^{-2}z; 0) + \chi(l; 2, 1 | q^{-2}z; 0) \right\} : . \end{aligned} \quad (3.4)$$

From (3.2), we then obtain

$$\begin{aligned} \Phi_{l,m}(z) &= \frac{1}{[m]!} \oint \frac{du_1}{2\pi i} \oint \frac{du_2}{2\pi i} \cdots \oint \frac{du_m}{2\pi i} \\ &\quad \times [\cdots [ [\Phi_{l,0}(z), J^+(u_1)]_{q^l}, J^+(u_2)]_{q^{l-2}} \cdots J^+(u_m)]_{q^{l-2(m-1)}}, \end{aligned} \quad (3.5)$$

for  $m = 1, 2, \dots, l$ .

Noting  $[\Phi_{l,m}(z), \tilde{a}_{\phi,0} + \tilde{a}_{\chi,0}] = 0$  and  $[\Phi_{l,m}(z), \eta_0] = 0$ , the vertex operator  $\Phi_{l,m}(z)$  implies a linear map  $\Phi_{l,m}(z) : W_{l'} \rightarrow W_{l'+l}$ . Further, noting the fact  $S(t) : W_l \rightarrow W_{l-2}$ , we obtain a linear map

$$\int_0^{c\infty} d_p t_1 \int_0^{c\infty} d_p t_2 \cdots \int_0^{c\infty} d_p t_r \Phi_{l,m}(z) S(t_1) S(t_2) \cdots S(t_r) : W_{l'} \rightarrow W_{l''}, \quad (3.6)$$

with  $2r = l' + l - l''$ . Since the screening charge  $\int_0^{c\infty} d_p t S(t)$  commutes with the operator  $\tilde{a}_{\phi,0} + \tilde{a}_{\chi,0}$ ,  $\eta_0$  and all the generators in  $U_q(\widehat{\mathfrak{sl}}_2)$ , we obtain the following identification:

$$\tilde{\Phi}_{\lambda_{l'},m}^{V^{(l)\lambda_{l''}}}(z) = g_{\lambda_{l'}}^{V^{(l)\lambda_{l''}}}(z) \Phi_{l,m}^{(r)}(z), \quad (3.7)$$

$$\Phi_{l,m}^{(r)}(z) = \int_0^{c\infty} d_p t_1 \int_0^{c\infty} d_p t_2 \cdots \int_0^{c\infty} d_p t_r \Phi_{l,m}(z) S(t_1) S(t_2) \cdots S(t_r). \quad (3.8)$$

Here  $g_{\lambda_{l'}}^{V^{(l)\lambda_{l''}}}(z)$  is the normalization factor to be determined in the below.

### 3.2 Cycles for the Jackson integral

The expression (3.8) is not well defined in the regions  $|t_i/z| > 1, i = 1, 2, \dots, r$ . We here determine suitable cycles, which make sense to the Jackson integrals in (3.8). Our argument is simply based on the requirement of the commutativity of the screening charge with the currents in  $U_q(\widehat{\mathfrak{sl}}_2)$ . Remarkably, it turns out that the resultant cycles imply the  $R$ -matrix

symmetry to the type II  $q$ -VOs. As will be shown in Sec.5, this symmetry is essential in the derivation of the properties of the form factors.

We have to consider the two cases  $\int d_p t \Phi_{l,0}(z)S(z)$  and  $\int d_p t S(t)\Phi_{l,0}(z)$ . As discussed in Ref[12], the only harmful operator in the commutativity with  $S(t)$  is the current  $J^-(w)$ , whose operator expansion with  $S(t)$  is given by

$$\begin{aligned} S(t)J^-(w) &= J^-(w)S(t) \\ &\sim {}_{k+2}\partial_t \left[ \frac{1}{w-t} : e^{-\Phi(k+2|q^{-2}t; \frac{k+2}{2})} : \right]. \end{aligned} \quad (3.9)$$

Therefore, in the commutation of  $\int d_p t \Phi_{l,0}(z)S(t)$  with  $J^-(w)$ , we have a Jackson integral of the total difference  ${}_{k+2}\partial_t$  of the function containing the factor

$$\Phi_{l,0}(z) : e^{-\Phi(k+2|q^{-2}t; \frac{k+2}{2})} : \sim \frac{(q^{l+2}t/z; p)_\infty}{(q^{-l+2}t/z; p)_\infty} : \Phi_{l,0}(z) e^{-\Phi(k+2|q^{-2}t; \frac{k+2}{2})} : . \quad (3.10)$$

In the same way, for  $\int d_p t S(t)\Phi_{l,0}(z)$ , we have the total difference of the function containing

$$: e^{-\Phi(k+2|q^{-2}t; \frac{k+2}{2})} : \Phi_{l,0}(z) \sim \frac{(q^{l+2k+2}z/t; p)_\infty}{(q^{-l+2k+2}z/t; p)_\infty} : \Phi_{l,0}(z) e^{-\Phi(k+2|q^{-2}t; \frac{k+2}{2})} : . \quad (3.11)$$

Now let us look for the finite regions, by which the Jackson integrals of the total differences vanish. By definition of the Jackson integral (2.35), it is easy to find that the Jackson integral of the total difference of (3.10) over any one of the regions  $[0, q^{k-l}p^{-j}z]$ ,  $j = 0, 1, 2, \dots$  vanishes. Similarly, we find that any one of the regions  $[q^{k+l}p^{j+1}z, q^{k+l}p^jz\infty)$ ,  $j = 0, 1, 2, \dots$  makes the total difference of (3.11) vanish.

Let us consider in detail the physically interesting case  $l = 1$ :

$$\tilde{\Phi}_\lambda^{V^{(1)\mu}}(z) = v_+ \otimes \tilde{\Phi}_{\lambda,0}^{V^{(1)\mu}}(z) + v_- \otimes \tilde{\Phi}_{\lambda,1}^{V^{(1)\mu}}(z), \quad (3.12)$$

with  $\mu = \lambda_\pm$ .

For  $\mu = \lambda_+$ , we have  $r = 0$  in (3.7), and obtain the expression

$$\tilde{\Phi}_{\lambda,0}^{V^{(1)\lambda_+}}(z) = g_\lambda^{V^{(1)\lambda_+}}(z)\Phi_{1,0}(z). \quad (3.13)$$

For  $\mu = \lambda_-$ , we have  $r = 1$  in (3.7) and obtain, from the above arguments, the following two expressions for the vertex  $\tilde{\Phi}_{\lambda,0}^{V^{(1)\lambda_-}}(z)$ :

$$g_\lambda^{V^{(1)\lambda_-}}(z) \int_0^{q^{k-1}z} d_p t \Phi_{1,0}(z)S(t), \quad (3.14)$$

$$\tilde{g}_\lambda^{V^{(1)\lambda_-}}(z) \int_{q^{k+1}pz}^{q^{k+1}z\infty} d_p t S(t)\Phi_{1,0}(z), \quad (3.15)$$

in the simplest choice  $j = 0$ .

The normalization functions can be easily calculated using the expression (3.4) and operator product expansion formulae in appendix A. From (2.28) and (3.1), we obtain the following results for the VOs  $\tilde{\Phi}_{\lambda_l}^{V^{(1)\lambda_{l+1}}}(z)$  and  $\tilde{\Phi}_{\lambda_l}^{V^{(1)\lambda_{l-1}}}(z)$ :

$$g_{\lambda_l}^{V^{(1)\lambda_{l+1}}}(z) = q^{-1-2l(k-2)s} z^{-2ls}, \quad (3.16)$$

$$g_{\lambda_l}^{V^{(1)\lambda_{l-1}}}(z) = -q^{2+l/2-2(3l+4)s} z^{(l+2)s} B_p(1-2ls, -2s)^{-1}. \quad (3.17)$$

Here  $s \equiv \frac{1}{2(k+2)}$ , and

$$B_p(x, y) = \int_0^1 d_p t \, t^{x-1} \frac{(pt; p)_\infty}{(p^y t; p)_\infty}$$

being the  $q$ -beta function[22]. It is useful to notice that in terms of the  $q$ -gamma function defined by

$$\Gamma_p(z) = \frac{(p; p)_\infty}{(p^z; p)_\infty} (1-p)^{1-z},$$

the  $q$ -beta function is expressed by

$$B_p(x, y) = \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)}.$$

In the same way, from (2.28), (3.1) and (3.15), we obtain

$$\tilde{g}_{\lambda_l}^{V^{(1)\lambda_{l-1}}}(z) = -q^{-1+l/2-2(l+3)s} z^{(l+2)s} B_p(2(l+1)s, -2s)^{-1}. \quad (3.18)$$

### 3.3 Commutation relation

We next discuss a commutation relation of the type II  $q$ -VOs. We show that the cycles obtained in the above lead to the  $R$ -matrix symmetry.

For this purpose, we calculate the following two point functions.

$$\langle \lambda | \tilde{\Phi}_{\mu}^{V^{(1)\lambda}}(z_2) \tilde{\Phi}_{\lambda}^{V^{(1)\mu}}(z_1) | \lambda \rangle = \sum_{\substack{m_1, m_2=0,1 \\ m_1+m_2=1}} v_{m_1}^{(1)} \otimes v_{m_2}^{(1)} \langle \lambda | \tilde{\Phi}_{\mu, m_2}^{V^{(1)\lambda}}(z_2) \tilde{\Phi}_{\lambda, m_1}^{V^{(1)\mu}}(z_1) | \lambda \rangle. \quad (3.19)$$

Note that in (3.19) only the cases  $\mu = \lambda_{\pm}$  give non-zero contributions. Let us first consider the case  $\mu = \lambda_{-}$ .

For the sake of brevity, let us set  $\Phi_{\lambda, +}^{\mu}(z) = \tilde{\Phi}_{\lambda, 0}^{V^{(1)\mu}}(z)$  and  $\Phi_{\lambda, -}^{\mu}(z) = \tilde{\Phi}_{\lambda, 1}^{V^{(1)\mu}}(z)$ . By the direct calculation using (3.14) for  $\Phi_{\lambda, +}^{\lambda_{-}}(z)$  and the operator expansion formulae given in Appendix A, we obtain the following results, for  $\lambda = \lambda_l$ .

$$\begin{aligned} & \langle \lambda | \Phi_{\lambda, +}^{\lambda_{-}}(z_2) \Phi_{\lambda, -}^{\lambda_{-}}(z_1) | \lambda \rangle / \xi(z; 1, q^4) \\ &= g_{\lambda_{-}}^{V^{(1)\lambda}}(z_2) g_{\lambda}^{V^{(1)\lambda_{-}}}(z_1) \int_0^{q^{k-1}z_1} d_p t \oint \frac{du}{2\pi i} \langle \lambda | \Phi_{1,0}(z_2) [\Phi_{1,0}(z_1), J^+(u)]_q S(t) | \lambda \rangle \\ &= B_p(1-2ls, -2s)^{-1} qz \int_0^1 d_p t t^{-2ls} \frac{(pt; p)_\infty (pzt; p)_\infty}{(q^{-2}pt; p)_\infty (q^{-2}zt; p)_\infty}, \end{aligned} \quad (3.20)$$

$$\begin{aligned}
& < \lambda | \Phi_{\lambda,-}^{\lambda}(z_2) \Phi_{\lambda,+}^{\lambda}(z_1) | \lambda > / \xi(z; 1, q^4) \\
& = g_{\lambda,-}^{V(1)\lambda}(z_2) g_{\lambda}^{V(1)\lambda-}(z_1) \int_0^{q^{k-1}z_1} d_p t \oint \frac{du}{2\pi i} < \lambda | [\Phi_{1,0}(z_2), J^+(u)]_q \Phi_{1,0}(z_1) S(t) | \lambda > \\
& = B_p(1-2ls, -2s)^{-1} \int_0^1 d_p t t^{-2ls} \frac{(pt; p)_{\infty} (zt; p)_{\infty}}{(q^{-2}t; p)_{\infty} (q^{-2}zt; p)_{\infty}}, \tag{3.21}
\end{aligned}$$

where  $z = z_1/z_2$ .

Here the function  $\xi(z; 1, q^4)$  is given by

$$\xi(z; a, b) = \frac{(az; p, q^4)_{\infty} (a^{-1}bz; p, q^4)_{\infty}}{(q^2az; p, q^4)_{\infty} (q^{-2}a^{-1}bz; p, q^4)_{\infty}}. \tag{3.22}$$

Combining (3.21) and (3.20), we obtain

$$\begin{aligned}
& < \lambda | \tilde{\Phi}_{\lambda,-}^{V_2^{(1)\lambda}}(z_2) \tilde{\Phi}_{\lambda}^{V_1^{(1)\lambda-}}(z_1) | \lambda > \\
& = F_p(1-2ls, 2s, 1-2(l+1)s; q^{-2}z) v_+ \otimes v_- \\
& + \frac{\Gamma_p(1-2(l+1)s) \Gamma_p(1-2s)}{\Gamma_p(2-2(l+1)s) \Gamma_p(-2s)} qz F_p(1-2ls, 1+2s, 2-2(l+1)s; q^{-2}z) v_- \otimes v_+ \tag{3.23}
\end{aligned}$$

Here we used the basic hypergeometric series  $F_p(a, b, c; : z)$  having the following Jackson integral presentation[22].

$$F_p(a, b, c; z) = \frac{\Gamma_p(c)}{\Gamma_p(a) \Gamma_p(c-a)} \int_0^1 d_p t t^{a-1} \frac{(pt; p)_{\infty} (p^b z t; p)_{\infty}}{(p^{c-a} t; p)_{\infty} (zt; p)_{\infty}}. \tag{3.24}$$

Next let us consider the case  $\mu = \lambda_+$ . In this case, using (3.15) for  $\tilde{\Phi}_{\lambda_+,+}^{V(1)\lambda}(z)$ , we obtain

$$\begin{aligned}
& < \lambda | \Phi_{\lambda_+,+}^{\lambda}(z_2) \Phi_{\lambda,-}^{\lambda}(z_1) | \lambda > / \xi(z; 1, q^4) \\
& = \tilde{g}_{\lambda_+}^{V(1)\lambda}(z_2) \tilde{g}_{\lambda}^{V(1)\lambda_+}(z_1) \int_{q^{k+1}pz_2}^{q^{k+1}z_2\infty} d_p t \oint \frac{du}{2\pi i} < \lambda | S(t) \Phi_{1,0}(z_2) [\Phi_{1,0}(z_1), J^+(u)]_q | \lambda > \\
& = B_p(2(l+2)s, -2s)^{-1} q^{-2} \int_p^{\infty} d_p t t^{-2(l+2)s-1} \frac{(p/t; p)_{\infty} (z/t; p)_{\infty}}{(q^{-2}/t; p)_{\infty} (q^{-2}z/t; p)_{\infty}} \\
& = B_p(2(l+2)s, -2s)^{-1} q^{-2} \int_0^1 d_p t t^{2(l+2)s-1} \frac{(pt; p)_{\infty} (zt; p)_{\infty}}{(q^{-2}t; p)_{\infty} (q^{-2}zt; p)_{\infty}}. \tag{3.25}
\end{aligned}$$

In the last line, we made a change  $t \rightarrow 1/t$  by the formula

$$\int_p^{\infty} d_p t f(t) = \int_0^1 d_p t t^{-2} f(1/t). \tag{3.26}$$

Similarly, we obtain

$$\begin{aligned}
& < \lambda | \Phi_{\lambda_+,+}^{\lambda}(z_2) \Phi_{\lambda_+,+}^{\lambda}(z_1) | \lambda > / \xi(z; 1, q^4) \\
& = \tilde{g}_{\lambda_+}^{V(1)\lambda}(z_2) \tilde{g}_{\lambda}^{V(1)\lambda_+}(z_1) \int_{q^{k+1}pz_2}^{q^{k+1}z_2\infty} d_p t \oint \frac{du}{2\pi i} < \lambda | S(t) [\Phi_{1,0}(z_2), J^+(u)]_q \Phi_{1,0}(z_1) | \lambda > \\
& = B_p(2(l+2)s, -2s)^{-1} q^{-1} \int_0^1 d_p t t^{2(l+2)s-1} \frac{(pt; p)_{\infty} (pzt; p)_{\infty}}{(q^{-2}pt; p)_{\infty} (q^{-2}zt; p)_{\infty}}. \tag{3.27}
\end{aligned}$$

Combining (3.25) and (3.27), we find

$$\begin{aligned}
& < \lambda | \tilde{\Phi}_{\lambda_+}^{V_2^{(1)\lambda}}(z_2) \tilde{\Phi}_{\lambda}^{V_1^{(1)\lambda_+}}(z_1) | \lambda > \\
& = F_p(2(l+2)s, 2s, 2(l+1)s; q^{-2}z) v_- \otimes v_+ \\
& + \frac{\Gamma_p(2(l+1)s)\Gamma_p(1-2s)}{\Gamma_p(1+2(l+1)s)\Gamma_p(-2s)} q F_p(2(l+2)s, 1+2s, 1+2(l+1)s; q^{-2}z) v_+ \otimes v_-. \quad (3.28)
\end{aligned}$$

The results (3.23) and (3.28) coincide with those obtained in Ref.[6], where no specific realizations of the  $q$ -VO's are used.

Using these results as well as the connection formula for the basic hypergeometric series[23] and the  $q$ -KZ equation for the two point function, one obtains the commutation relation of the type II  $q$ -VO (the  $R$ -matrix symmetry)[6]:

$$\Phi_{\lambda_{\pm}, \varepsilon_1}^{\nu}(z_1) \Phi_{\lambda, \varepsilon_2}^{\lambda_{\pm}}(z_2) = R_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2}(z) \sum_{\mu=\lambda_+, \lambda_-} \Phi_{\mu, \varepsilon'_2}^{\lambda}(z_2) \Phi_{\lambda, \varepsilon'_1}^{\mu}(z_1) W \left( \begin{matrix} \lambda & \lambda_{\pm} \\ \mu & \nu \end{matrix} \middle| z \right), \quad (3.29)$$

where  $\varepsilon_i = \pm, i = 1, 2$ , and the coefficients  $R_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2}(z)$  are the  $R$ -matrix,  $R(z) : V^{(1)} \otimes V^{(1)} \rightarrow V^{(1)} \otimes V^{(1)}$  given by

$$R_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2}(z) = r(z) r_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2}(z), \quad r(z) = z^{-1/2} \frac{(q^4/z; q^4)_{\infty} (q^2 z; q^4)_{\infty}}{(q^4 z; q^4)_{\infty} (q^2/z; q^4)_{\infty}}, \quad (3.30)$$

$$r_{++}^{++}(z) = r_{--}^{--}(z) = 1, \quad (3.31)$$

$$r_{+-}^{+-}(z) = r_{-+}^{-+}(z) = \frac{1-z}{1-q^2 z} q, \quad (3.32)$$

$$r_{+-}^{-+}(z) = z r_{-+}^{+-}(z) = \frac{1-q^2}{1-q^2 z} z. \quad (3.33)$$

The factors  $W \left( \begin{matrix} \lambda & \mu \\ \mu' & \nu \end{matrix} \middle| z \right)$  are given by

$$W \left( \begin{matrix} \lambda & \mu \\ \mu' & \nu \end{matrix} \middle| z \right) = -z^{\Delta_{\lambda} + \Delta_{\nu} - \Delta_{\mu} - \Delta_{\mu'} - 1/2} \frac{\xi(z^{-1}; 1, pq^4)}{\xi(z; 1, pq^4)} \hat{W} \left( \begin{matrix} \lambda & \mu \\ \mu' & \nu \end{matrix} \middle| z \right), \quad (3.34)$$

$$\hat{W} \left( \begin{matrix} \lambda & \lambda_+ \\ \lambda_+ & \lambda \end{matrix} \middle| z \right) = \frac{\Theta_p(pq^2)}{\Theta_p(pq^{-2l-2})} \frac{\Theta_p(pq^{-2l-2}z)}{\Theta_p(pq^2 z)}, \quad (3.35)$$

$$\hat{W} \left( \begin{matrix} \lambda & \lambda_+ \\ \lambda_- & \lambda \end{matrix} \middle| z \right) = q^{-1} \frac{\Gamma_p((2l+2)s)^2}{\Gamma_p((2l+4)s)\Gamma_p(2ls)} \frac{\Theta_p(pz)}{\Theta_p(pq^2 z)}, \quad (3.36)$$

$$\hat{W} \left( \begin{matrix} \lambda & \lambda_- \\ \lambda_+ & \lambda \end{matrix} \middle| z \right) = q^{-1} \frac{\Gamma_p(1-(2l+2)s)^2}{\Gamma_p(1-(2l+4)s)\Gamma_p(1-2ls)} \frac{\Theta_p(pz)}{\Theta_p(pq^2 z)}, \quad (3.37)$$

$$\hat{W} \left( \begin{matrix} \lambda & \lambda_- \\ \lambda_- & \lambda \end{matrix} \middle| z \right) = z^{-1} \frac{\Theta_p(pq^2)}{\Theta_p(q^{2l+2})} \frac{\Theta_p(q^{2l+2}z)}{\Theta_p(pq^2 z)}, \quad (3.38)$$

$$\hat{W} \left( \begin{matrix} \lambda & \lambda_{\pm} \\ \lambda_{\pm} & \lambda_{\pm} \end{matrix} \middle| z \right) = 1, \quad (3.39)$$

$$\hat{W} \left( \begin{matrix} \lambda & \mu \\ \mu' & \nu \end{matrix} \middle| z \right) = 0 \quad \text{otherwise.} \quad (3.40)$$



Here  $\Theta_p(z) = (z; p)_\infty (z^{-1}p; p)_\infty (p; p)_\infty$ . These factors can be regarded as the Boltzmann weight in the face formulation.

## 4 Integral Formula for Form Factors

In this section, we evaluate a trace of the  $q$ -VOs over IHWR of  $U_q(\widehat{\mathfrak{sl}}_2)$  and derive a general integral formula for form factors in the spin  $k/2$  antiferromagnetic XXZ model. We here assume  $-1 < q < 0$ [4].

Let  $\mathcal{L}_\lambda$  be a local operator acting on the  $q$ -Wakimoto module  $W_\lambda$ . As discussed in Ref.[4, 6], the operator  $\mathcal{L}_\lambda$  is given by the product of the type I  $q$ -vertex operators. It enjoys the property

$$\mathcal{L}_\lambda \Phi_\mu^{V^{(1)\lambda}}(z) = \Phi_\mu^{V^{(1)\lambda}}(z) \mathcal{L}_\mu. \quad (4.1)$$

In the following, we do not specify any form for  $\mathcal{L}_\lambda$ .

The  $N$  particle form factor of the local operator  $\mathcal{L}_\lambda$  is then given by the following trace[4, 6].

$$\begin{aligned} & F_{\lambda\lambda_{N-1}\lambda_{N-2}\dots\lambda_1\lambda}(\zeta_N, \zeta_{N-1}, \dots, \zeta_1) \\ &= \frac{\text{Tr}_{\mathcal{H}_\lambda} \left( q^{4\tilde{L}_0} q^{-J_0^3} \mathcal{L}_\lambda \Phi_{\lambda_{N-1}}^{V^*\lambda}(\zeta_N) \Phi_{\lambda_{N-2}}^{V^*\lambda_{N-1}}(\zeta_{N-1}) \cdots \Phi_{\lambda_1}^{V^*\lambda_1}(\zeta_1) \right)}{\text{Tr}_{\mathcal{H}_\lambda} \left( q^{4\tilde{L}_0} q^{-J_0^3} \right)}, \end{aligned} \quad (4.2)$$

where  $\tilde{L}_0 \equiv L_0 - \frac{c}{24}$  and  $c \equiv \frac{3k}{k+2}$ . The vertex  $\Phi_\mu^{V^*\lambda}(\zeta)$  is an anti-pole dual of the type II  $q$ -vertex operator  $\Phi_\mu^{V^{(1)\lambda}}(z)$ , describing the creation of physical particles with spin  $1/2$ [6]. It is related to the type II  $q$ -VO as follows.

Let us expand the vertex  $\Phi_\mu^{V^*\lambda}(\zeta)$  as

$$\Phi_\mu^{V^*\lambda}(\zeta) = v_+ \otimes \Phi_{\mu,+}^{*\lambda}(\zeta) + v_- \otimes \Phi_{\mu,-}^{*\lambda}(\zeta), \quad (4.3)$$

then the following relations are held[6].

$$\Phi_{\lambda,-}^{*\lambda+}(\zeta) = -q \Phi_{\lambda,+}^{\lambda+}(q^2\zeta), \quad (4.4)$$

$$\Phi_{\lambda,+}^{*\lambda+}(\zeta) = \Phi_{\lambda,-}^{\lambda+}(q^2\zeta), \quad (4.5)$$

$$\Phi_{\lambda,-}^{*\lambda-}(\zeta) = \Phi_{\lambda,+}^{\lambda-}(q^2\zeta), \quad (4.6)$$

$$\Phi_{\lambda,+}^{*\lambda-}(\zeta) = -q^{-1} \Phi_{\lambda,-}^{\lambda-}(q^2\zeta). \quad (4.7)$$

Note that using the  $R$ -matrix symmetry (3.29) and the cyclic property of trace, one can easily show that the form factor (4.2) satisfy the  $q$ -KZ equation.

Now, let us evaluate the traces. The calculation can be done by using the coherent state method, which is familiar in superstring theory. For example, see Ref.[24].

Let us first consider the normalization factor  $\text{Tr}_{\mathcal{H}_\lambda}(q^{4\tilde{L}_0}q^{-J_0^3})$ . This is essentially the character of  $\mathcal{H}_\lambda$ . As evaluated in Ref.[12], it coincides with the classical one, i.e., the Weyl-Kac formula for  $\widehat{\mathfrak{sl}}_2$ .

$$\begin{aligned} & \text{Tr}_{\mathcal{H}_\lambda}(q^{4\tilde{L}_0}q^{-J_0^3}) \\ &= \frac{1}{\vartheta_1(\tilde{q}^2|\tau)} \sum_{s \in \mathbf{Z}} \left[ q^{4PP'(s - \frac{nP' - n'P}{2PP'})^2} \tilde{q}^{2P(s - \frac{nP' - n'P}{2PP'})} - q^{4PP'(s + \frac{nP' + n'P}{2PP'})^2} \tilde{q}^{2P(s + \frac{nP' + n'P}{2PP'})} \right], \end{aligned} \quad (4.8)$$

for  $\lambda = \lambda_{l_{n,n'}}$ , where  $\tau \equiv \frac{\log \zeta}{2\pi i}$ , and  $\tilde{q} \equiv q^{-\frac{2 \log q}{q - q^{-1}}}$ .

Next, let us consider the numerator. From the trace formula (2.39) and (4.4)  $\sim$  (4.7), it is given by a combination of the following traces

$$\begin{aligned} & \text{Tr}_{W_{\lambda_l}} \left( \zeta^{\tilde{L}_0} y^{-J_0^3} \prod_{i=1}^a J^-(w_i) \prod_{r=1}^m \Psi_{g,g}(\zeta_r) \prod_{s=1}^n \Phi_{1,0}(z_s) \prod_{j=1}^b S(t_j) \prod_{h=1}^c J^+(u_h) \right) \\ &= \sum_{\{\epsilon\}, \{\delta\}, \{\rho\}} \frac{(-)^{b+c} \prod \epsilon_i \prod \delta_j \prod \rho_h}{(q - q^{-1})^{a+b+c} \prod w_i \prod t_j \prod u_h} f_\Phi^{\lambda_l}(\{\epsilon\}) f_{\phi\chi}(\{\epsilon\}, \{\delta\}, \{\rho\}). \end{aligned} \quad (4.9)$$

Here we split the operators  $J^\pm(w)$  and  $S(t)$  as follows.

$$J^-(z) = \frac{1}{(q - q^{-1})z} \sum_{\epsilon=\pm 1} \epsilon J_\epsilon^-(z), \quad (4.10)$$

$$J^+(z) = -\frac{1}{(q - q^{-1})z} \sum_{\rho=\pm 1} \rho J_\rho^+(z), \quad (4.11)$$

$$S(z) = -\frac{1}{(q - q^{-1})z} \sum_{\delta=\pm 1} \delta S_\delta(z), \quad (4.12)$$

where

$$\begin{aligned} J_\epsilon^-(z) &= : \exp \left\{ \partial \Phi^{(\epsilon)}(q^{-2}z; -\frac{k+2}{2}) \right. \\ &\quad \left. + \phi(2 | q^{(\epsilon-1)(k+2)}z; -1) + \chi(2 | q^{(\epsilon-1)(k+1)-1}z; 0) \right\} :, \end{aligned} \quad (4.13)$$

$$J_\rho^+(z) = : \exp \left\{ -\phi(2 | q^{-k-2}z; 1) - \chi(2 | q^{-k-2+\rho}z; 0) \right\} :, \quad (4.14)$$

$$\begin{aligned} S_\delta(z) &= : \exp \left\{ -\Phi(k+2 | q^{-2}z; -\frac{k+2}{2}) \right. \\ &\quad \left. - \phi(2 | q^{-k-2}z; -1) - \chi(2 | q^{-k-2+\delta}z; 0) \right\} :, \end{aligned} \quad (4.15)$$

In (4.13),  $\partial \Phi^{(\epsilon)}(q^{-2}z; -\frac{k+2}{2})$  denotes the field

$$\partial \Phi^{(\epsilon)}(q^{-2}z; -\frac{k+2}{2}) = \epsilon \left\{ (q - q^{-1}) \sum_{n=1}^{\infty} a_{\Phi, \epsilon n} z^{-\epsilon n} q^{(2\epsilon - \frac{k+2}{2})n} + \tilde{a}_{\Phi, 0} \log q \right\}.$$

The factors  $f_\Phi^{\lambda_l}(\{\epsilon\})$  and  $f_{\phi\chi}(\{\epsilon\}, \{\delta\}, \{\rho\})$  are the contributions from the  $\Phi$  and the  $\phi\chi$  sector, respectively:

$$f_\Phi^{\lambda_l}(\{\epsilon\}) = \text{Tr}_{F_l^\Phi} \left( \zeta^{L_0^\Phi - \frac{c-2}{24}} y^{-\tilde{a}_{\Phi, 0}} \prod_{i=1}^a e^{\partial \Phi^{(\epsilon)}(q^{-2}w_i; -\frac{k+2}{2})} \prod_{r=1}^m e^{\Phi(g; 2, k+2 | q^k \zeta_r; \frac{k+2}{2})} \right)$$

$$\times \prod_{s=1}^n e^{\Phi(1;2,k+2|q^{k-1}z_s;-\frac{k+2}{2})} \prod_{j=1}^b e^{-\Phi(k+2|q^{-2}t_j;-\frac{k+2}{2})} \Big), \quad (4.16)$$

$$\begin{aligned} f_{\phi\chi}(\{\epsilon\}, \{\delta\}, \{\rho\}) &= \oint \frac{dw_0}{2\pi i w_0} \oint \frac{dw}{2\pi i} \text{Tr}_{F\phi\chi} \left( \zeta^{L_0^\phi + L_0^\chi - \frac{1}{12}} y^{-\tilde{a}_{\phi,0}} \prod_{i=1}^a e^{\phi(2|\tilde{W}_i^\epsilon;-1)} \prod_{s=1}^n e^{\phi(2|Z_s;0)} \right. \\ &\quad \times \prod_{j=1}^b e^{-\phi(2|T_j;-1)} \prod_{j=1}^b e^{-\phi(2|U_h;1)} e^{\chi(2|q^{-k-2}w;0)} \prod_{i=1}^a e^{\chi(2|W_i^\epsilon;0)} \\ &\quad \left. \times \prod_{s=1}^n e^{\chi(2|Z_s;0)} e^{-\chi(2|W_0;0)} \prod_{j=1}^b e^{-\chi(2|T_j^\delta;0)} \prod_{j=1}^b e^{-\chi(2|U_h^\rho;0)} \right). \quad (4.17) \end{aligned}$$

Here we have set  $T_j \equiv q^{-k-2}t_j$ ,  $U_h \equiv q^{-k-2}u_h$ ,  $W_0 \equiv q^{-k-2}w_0$ ,  $T_j^\delta \equiv q^{\delta_j}T_j$ ,  $U_h^\rho \equiv q^{\rho_h}U_h$ ,  $Z_s \equiv q^{-2}z_s$ ,  $W_i^\epsilon \equiv q^{(\epsilon_i-1)(k+1)-1}w_i$ , and  $\tilde{W}_i^\epsilon \equiv q^{(\epsilon_i-1)(k+2)}w_i$ .

Now we evaluate the factors  $f_\Phi^{\lambda_l}(\{\epsilon\})$  and  $f_{\phi\chi}(\{\epsilon\}, \{\delta\}, \{\rho\})$  separately. For the  $\Phi$  sector, we obtain the following result:

$$\begin{aligned} f_\Phi^{\lambda_l}(\{\epsilon\}) &= \eta(\tau)^{-1} \zeta^{\frac{l(l+2)+1}{4(k+2)}} y^{-l} q^{l\sum \epsilon_i} \left[ \frac{\prod (q^k \zeta_r)^g \prod q^{k-2} z_s}{\prod (q^{-2} t_j)^2} \right]^{l/2(k+2)} \\ &\quad \times \prod_{j < j'} G_S(t_j, t_{j'}) \prod_{s < s'} G_\Phi(z_s, z_{s'}) \prod_{j,s} G_{\Phi S}(t_j, z_s) \\ &\quad \times \prod_{i < i'} G_{J-}(w_i, w_{i'}) \prod_{r < r'} G_\Psi(\zeta_r, \zeta_{r'}) \prod_{j,i} G_{SJ-}(t_j, w_i) \\ &\quad \times \prod_{j,r} G_{S\Psi}(t_j, \zeta_r) \prod_{i,r} G_{J-\Psi}(w_j, \zeta_r) \prod_{r,s} G_{\Psi\Phi}(\zeta_r, z_s) \prod_{i,s} G_{J-\Phi}(w_j, z_s), \quad (4.18) \end{aligned}$$

where the function  $\eta(\tau)$  is Dedekind's  $\eta$ -function

$$\eta(\tau) = \zeta^{1/24} (\zeta; \zeta)_\infty.$$

The expressions of the two point functions  $G_S(t, t')$  etc., are given in Appendix B.

For the  $\phi\chi$  sector, we have a master formula ((56) in Ref.[12]). Applying that formula to the case (4.17), we obtain the results.

$$\begin{aligned} f_{\phi\chi}(\{\epsilon\}, \{\delta\}, \{\rho\}) &= \eta(\tau) y^{-1} q^{\sum \epsilon} (q^{-2}\zeta; \zeta)_\infty^{-a-b} (q^2\zeta; \zeta)_\infty^{-c} (\zeta; \zeta)_\infty^{-n} \\ &\quad \times \oint \frac{dw_0}{2\pi i w_0} \frac{\prod_1^b E(W_0, T_j^\delta) \prod_1^c E(W_0, U_h^\rho)}{\prod_1^a E(W_i^\epsilon, W_0) \prod_1^n E(Z_s, W_0)} \prod_{j,h} \frac{E(T_j^\delta, U_h^\rho)}{E(T_j, U_h)} \prod_{i,h} \frac{E(\tilde{W}_i^\epsilon, U_h)}{E(W_i^\epsilon, U_h^\rho)} \\ &\quad \times \prod_{i < i'} \frac{E(W_i^\epsilon, W_{i'}^\epsilon)}{E_q(\tilde{W}_i^\epsilon, \tilde{W}_{i'}^\epsilon; -2)} \prod_{j < j'} \frac{E(T_j^\delta, T_{j'}^\delta)}{E_q(T_j, T_{j'}; -2)} \prod_{h < h'} \frac{E(U_h^\rho, U_{h'}^\rho)}{E_q(U_h, U_{h'}; 2)} \\ &\quad \times \prod_{i,s} \frac{E(W_i^\epsilon, Z_s)}{E_q(\tilde{W}_i^\epsilon, Z_s; -1)} \prod_{i,j} \frac{E_q(\tilde{W}_i^\epsilon, T_j^\delta; -2)}{E(W_i^\epsilon, T_j^\delta)} \prod_{i,j} \frac{E_q(Z_s, T_j; -1)}{E(Z_s, T_j^\delta)} \prod_{s,h} \frac{E_q(Z_s, U_h; 1)}{E(Z_s, U_h^\rho)} \end{aligned}$$

$$\begin{aligned}
& \times \frac{\prod_{i=1}^a \vartheta_1(W_i^\epsilon + q^{-\sum \epsilon - \sum \delta - \sum \rho} - W_0 + y^2 | \tau)}{\prod_{j=1}^b \vartheta_1(T_j^\delta + q^{-\sum \epsilon - \sum \delta - \sum \rho} - W_0 + y^2 | \tau)} \\
& \times \frac{\prod_{s=1}^n \vartheta_1(Z_s + q^{-\sum \epsilon - \sum \delta - \sum \rho} - W_0 + y^2 | \tau)}{\vartheta_1(q^{-\sum \epsilon - \sum \delta - \sum \rho} + y^2 | \tau) \prod_{h=1}^c \vartheta_1(U_h^\rho + q^{-\sum \epsilon - \sum \delta - \sum \rho} - W_0 + y^2 | \tau)}. \quad (4.19)
\end{aligned}$$

Here we used the abridged notations  $T_i$  to express  $\frac{\log T_i}{2\pi i}$  etc. in theta functions. The functions  $E(w, z)$  and  $E_q(w, z; \alpha)$  are given by

$$E(w, z) = \frac{1}{\sqrt{z/w}} \frac{(z/w; \zeta)_\infty (\zeta w/z; \zeta)_\infty}{(\zeta; \zeta)_\infty^2}, \quad (4.20)$$

$$E_q(w, z; \alpha) = \frac{1}{\sqrt{z/w}} (q^\alpha z/w; \zeta)_\infty (q^\alpha \zeta w/z; \zeta)_\infty. \quad (4.21)$$

## 5 Residue Formula

In this section, we analyze the pole structures of the form factors. We then derive the formulae giving the residues of corresponding poles. The derivation given here is independent of the integral formulae obtained in the last section. In Appendix C, we give another derivation based on the integral formulae.

Let us consider the following coefficients.

$$\begin{aligned}
& f_{\lambda \lambda_{N-1} \lambda_{N-2} \dots \lambda_2 \lambda_1 \lambda}^{\varepsilon_N \varepsilon_{N-1} \dots \varepsilon_2 \varepsilon_1}(z_N, z_{N-1}, \dots, z_2, z_1) \\
& = \text{Tr}_{\mathcal{H}_\lambda} \left( q^{4\tilde{L}_0} q^{-J_0^3} \mathcal{L}_\lambda \Phi_{\lambda_{N-1}, \varepsilon_N}^\lambda(z_N) \Phi_{\lambda_{N-2}, \varepsilon_{N-1}}^{\lambda_{N-1}}(z_{N-1}) \dots \Phi_{\lambda_1, \varepsilon_2}^{\lambda_2}(z_2) \Phi_{\lambda, \varepsilon_1}^{\lambda_1}(z_1) \right). \quad (5.1)
\end{aligned}$$

Let us investigate a local behavior of the product  $\Phi_{\lambda_1, \varepsilon_2}^{\lambda_2}(z_2) \Phi_{\lambda, \varepsilon_1}^{\lambda_1}(z_1)$ . Since the  $q$ -VOs are intertwiner of  $U_q(\widehat{\mathfrak{sl}}_2)$ , in the trace over IHWR, properties of any matrix elements of the product are determined by those taken by the highest weight states  $|\lambda\rangle_s[4]$ . We therefore only have to investigate the pole structure of the two point functions calculated in Sec.3.

For example, let us consider the function (3.20). Using the definition of the Jackson integral (2.35), one can show that a simple pole at  $q^{-2}z_1/z_2 = 1$  appears from the first term in the infinite series. One thus obtains the following local structures at the neighborhood of the point  $q^{-2}z_1/z_2 = 1$ :

$$\Phi_{\lambda_-, +}^\mu(z_2) \Phi_{\lambda_-, -}^{\lambda_-}(z_1) = q^3(1-p) \frac{(q^2 p; p)_\infty}{(q^{-2} p; p)_\infty} \frac{\xi(q^2; 1, q^4)}{B_p(1-2ls, -2s)} \delta_\lambda^\mu \frac{1}{1 - q^{-2}z_1/z_2} \text{id} + O(1).$$

In the same analysis for the remaining two point functions, one obtains

$$\Phi_{\lambda_\pm, \varepsilon_2}^\mu(z_2) \Phi_{\lambda, \varepsilon_1}^{\lambda_\pm}(z_1) = \mathcal{N}_\lambda^{\lambda_\pm}(\varepsilon_2, \varepsilon_1) \delta_{\varepsilon_2, -\varepsilon_1} \delta_\lambda^\mu \frac{1}{1 - q^{-2}z_1/z_2} \text{id} + O(1), \quad (5.2)$$

where

$$\begin{aligned}\mathcal{N}_\lambda^{\lambda-}(+, -) &= -q^4(1-p) \frac{(q^2p; p)_\infty}{(q^{-2}p; p)_\infty} \frac{\xi(q^2; 1, q^4)}{B_p(1-2ls, -2s)} \\ &= -q\mathcal{N}_{\lambda-}(\lambda; -, +),\end{aligned}\tag{5.3}$$

$$\begin{aligned}\mathcal{N}_\lambda^{\lambda+}(+, -) &= q(1-p) \frac{(q^2p; p)_\infty}{(q^{-2}p; p)_\infty} \frac{\xi(q^2; 1, q^4)}{B_p(2(l+2)s, -2s)} \\ &= -q\mathcal{N}_\lambda^{\lambda+}(-, +).\end{aligned}\tag{5.4}$$

Generally, in taking the trace  $\text{Tr}_{\mathcal{H}_\lambda}(\zeta^{L_0} \dots \Phi_{\lambda_\pm, \varepsilon_2}^\mu(z_2) \Phi_{\lambda, \varepsilon_1}^{\lambda_\pm}(z_1))$ , one must deal with additional poles. Namely, using the relation

$$\zeta^{L_0} \Phi_{\lambda, \varepsilon}^\mu(z) = \Phi_{\lambda, \varepsilon}^\mu(\zeta z) \zeta^{L_0}\tag{5.5}$$

and the cyclic property of trace, one can obtain the trace  $\text{Tr}_{\mathcal{H}_\rho}(\zeta^{L_0} \dots \Phi_{\lambda, \varepsilon_1}^{\lambda_\pm}(z_1) \Phi_{\rho, \varepsilon_2}^\lambda(\zeta z_2))$ . Therefore, one has to take into account the poles arising from

$$\Phi_{\rho_\pm, \varepsilon_1}^\sigma(z_1) \Phi_{\rho, \varepsilon_2}^{\rho_\pm}(\zeta z_2) = \mathcal{N}_\rho^{\rho_\pm}(\varepsilon_1, \varepsilon_2) \delta_{\varepsilon_2, -\varepsilon_1} \delta_\rho^\sigma \frac{1}{1 - q^{-2} \zeta z_2 / z_1} \text{id} + O(1).\tag{5.6}$$

In our case,  $\zeta = q^4$ . One hence has to consider the both contributions from (5.2) and (5.6) in the calculation of the residue at  $q^{-2} z_1 / z_2 = 1$ .

Noting these facts, as well as the commutation relation of the type II  $q$ -VOs (3.29), we finally obtain the following residue

$$\begin{aligned}q^{-2} z_1 / z_2 &= 1 \text{ res } f_{\lambda \lambda_{N-1} \dots \lambda_1 \lambda}^{\varepsilon_N \varepsilon_{N-1} \dots \varepsilon_1}(z_N, z_{N-1}, \dots, z_1) \\ &= \delta_\lambda^{\lambda_2} \delta_{\varepsilon_2, -\varepsilon_1} \mathcal{N}_\lambda^{\lambda_1}(-\varepsilon_1, \varepsilon_1) f_{\lambda \lambda_{N-1} \dots \lambda_3 \lambda}^{\varepsilon_N \varepsilon_{N-1} \dots \varepsilon_3}(z_N, z_{N-1}, \dots, z_3) \\ &\quad - \mathcal{N}_{\lambda_1}^\lambda(\varepsilon_1, -\varepsilon_1) \delta_{\tau_{N-2}, -\varepsilon_1} R_{\varepsilon_N \tau_{N-3}}^{\rho_N \tau_{N-2}}\left(\frac{z_N}{z_2}\right) R_{\varepsilon_{N-1} \tau_{N-4}}^{\rho_{N-1} \tau_{N-3}}\left(\frac{z_{N-1}}{z_2}\right) \dots R_{\varepsilon_3 \tau_1}^{\rho_4 \tau_2}\left(\frac{z_4}{z_2}\right) R_{\varepsilon_3 \varepsilon_2}^{\rho_3 \tau_1}\left(\frac{z_3}{z_2}\right) \\ &\quad \times \sum_{\mu_1, \dots, \mu_{N-3}} W\left(\begin{matrix} \mu_{N-3} & \lambda_{N-1} \\ \lambda_1 & \lambda \end{matrix} \middle| \frac{z_N}{z_2}\right) W\left(\begin{matrix} \mu_{N-4} & \lambda_{N-2} \\ \mu_{N-3} & \lambda_{N-1} \end{matrix} \middle| \frac{z_N}{z_2}\right) \dots W\left(\begin{matrix} \mu_1 & \lambda_3 \\ \mu_2 & \lambda_4 \end{matrix} \middle| \frac{z_4}{z_2}\right) W\left(\begin{matrix} \lambda_1 & \lambda_2 \\ \mu_1 & \lambda_3 \end{matrix} \middle| \frac{z_3}{z_2}\right) \\ &\quad \times f_{\lambda_1 \mu_{N-3} \dots \mu_1 \lambda_1}^{\rho_N \rho_{N-1} \dots \rho_3}(z_N, z_{N-1}, \dots, z_3).\end{aligned}\tag{5.7}$$

This result satisfies all the characteristic features of the Smirnov's third axiom. Namely, the form factor has only a simple pole, and the residue of the  $N$ -particle form factor is given by the  $N-2$ -particle form factors as well as by the  $R$ -matrix elements. In addition to these features, our result shows that the form factor in the higher spin XXZ model requires face type Boltzmann weights  $W\left(\begin{matrix} \lambda & \mu \\ \mu' & \nu \end{matrix} \middle| z\right)$ .

## 6 Conclusion

We have obtained a free field realization of the type II  $q$ -vertex operators of  $U_q(\widehat{\mathfrak{sl}}_2)$  of arbitrary level  $k$ . Determining the cycles for the Jackson integral associated with the screening charges, we have showed that our type II  $q$ -vertex operators satisfies the  $R$ -matrix symmetry. We have also derived an integral formulae for form factors in the spin  $k/2$  XXZ model. It has been shown that the result has a proper analytic structure and satisfies the higher spin extension of the lattice analog of the Smirnov's third axiom. Combining this result with those obtained in Ref.[6], we now recover all the data necessary to define the Riemann-Hilbert problem for form factors.

Since our form factor is constructed fully based on the representation theory of  $U_q(\widehat{\mathfrak{sl}}_2)$ , it is likely that the form factors in the massive integrable quantum field theory ( e.g. higher spin extension of the  $SU(2)$  invariant Thirring model, which is expected to be a certain scaling limit of our lattice model ) can also be understood in terms of the representation theory of quantum algebra. For this purpose, to investigate the scaling limit of the form factors is an interesting problem[25].

On the other hand, recently the idea of “vertex operator construction of form factors” has been applied by Lukyanov directly to the  $SU(2)$  invariant Thirring model as well as the sine-Gordon model[26]. There, the Fadeev-Zamolodchikov generators, which correspond to our type II  $q$ -vertex operators, were realized by means of free fields. However, the expected underlying infinite dimensional symmetry, i.e. Yangian, and the representation theoretical meaning of his vertex operators, i.e. intertwining property, are still unclear. To make a connection between Lukyanov's results with those obtained in Ref.[4, 5] and this paper could be helpful to clarify these points.

We hope to discuss these subject in the future.

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## A Operator Product Expansion Formulae

We here summarize the operator product expansion formulae among the current  $J^+(z)$ , the screening operator  $S(z)$  and the  $q$ -vertex operator  $\Phi_{l,0}(z)$ .

By splitting the operators  $J^+(u)$  and  $S(t)$  as (4.14) and (4.15), we obtain as follows.

$$\begin{aligned}
J_1^+(u)\Phi_{l,0}(z) &= q^{-l}\frac{u-q^{k+l}z}{u-q^{k-l}z} : J_1^+(u)\Phi_{l,0}(z) : & |u| > q^{k-l}|z| \\
J_{-1}^+(u)\Phi_{l,0}(z) &= q^l : J_{-1}^+(z)\Phi_{l,0}(z) : \\
\Phi_{l,0}(z)J_1^+(u) &= : \Phi_{l,0}(z)J_1^+(u) :
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\Phi_{l,0}(z)J_{-1}^+(u) &= \frac{z-q^{l-k}u}{z-q^{-l-k}u} : \Phi_{l,0}(w)J_{-1}^+(z) : & |z| > q^{-l-k}|u| \\
S_1(t)J_1^+(u) &= q : J_1^+(u)S_1(t) : \\
S_1(t)J_{-1}^+(u) &= q\frac{t-q^{-2}u}{t-u} : J_{-1}^+(u)S_1(t) : & |t| > q^{-2}|u| \\
S_{-1}(t)J_1^+(u) &= q^{-1}\frac{t-q^2z}{t-u} : J_1^+(u)S_{-1}(t) : & |t| > |u| \\
S_{-1}(t)J_{-1}^+(u) &= q^{-1} : J_{-1}^+(u)S_{-1}(t) :
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
J_1^+(u)S_1(t) &= q : J_1^+(u)S_1(t) : \\
J_1^+(u)S_{-1}(t) &= q\frac{u-q^{-2}t}{u-t} : J_1^+(u)S_{-1}(t) : & |u| > q^{-2}|t| \\
J_{-1}^+(u)S_1(t) &= q^{-1}\frac{u-q^2t}{u-t} : J_{-1}^+(u)S_1(t) : & |u| > |t| \\
J_{-1}^+(u)S_{-1}(t) &= q^{-1} : J_{-1}^+(u)S_{-1}(t) :
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
S_1(t)\Phi_{1,0}(z) &= (q^{-2}t)^{-1/k+2}q^{-1}\frac{(q^{k+1}z/t;p)_\infty}{(q^{k-1}z/t;p)_\infty} : S_1(t)\Phi_{1,0}(z) : & |t| > q^{k-1}|z| \\
S_{-1}(t)\Phi_{1,0}(z) &= (q^{-2}t)^{-1/k+2}q\frac{(q^{k+1}pz/t;p)_\infty}{(q^{k-1}pz/t;p)_\infty} : S_{-1}(t)\Phi_{1,0}(z) : & |t| > q^{k-1}p|z| \\
\Phi_{1,0}(z)S_1(t) &= (q^{k-2}z)^{-1/k+2}\frac{(q^{-k+1}pt/z;p)_\infty}{(q^{-k-1}pt/z;p)_\infty} : S_1(z)\Phi_{1,0}(z) : & |z| > q^{-k-1}p|t| \\
\Phi_{1,0}(z)S_{-1}(t) &= (q^{k-2}z)^{-1/k+2}\frac{(q^{-k+1}t/z;p)_\infty}{(q^{-k-1}t/z;p)_\infty} : S_{-1}(t)\Phi_{1,0}(z) : & |z| > q^{-k-1}|t| \\
\Phi_{l,0}(z)\Phi_{l,0}(w) &= (q^{k-2}z)^{l^2/2(k+2)}\frac{(q^{2(1-l)}w/z;p,q^4)_\infty(q^{2(1+l)}w/z;p,q^4)_\infty}{(q^2w/z;p,q^4)_\infty^2} : \Phi_{l,0}(z)\Phi_{l,0}(w) :
\end{aligned} \tag{A.4}$$

Especially for  $l = 1$ , we have

$$\Phi_{1,0}(z)\Phi_{1,0}(w) = \xi(w/z; 1, q^4) : \Phi_{1,0}(z)\Phi_{1,0}(w) : . \tag{A.6}$$

## B Two Point Functions

We here give the expressions only for  $G_S(t, u)$ ,  $G_{S\Phi}(t, z)$ ,  $G_{\Phi S}(z, t)$  and  $G_\Phi(z, w)$ , which are used in the text.

$$G_S(t, u) = \mathcal{N}_S (q^{-2}t)^{2/k+2} \frac{(q^{-2}u/t; p, \zeta)_\infty (q^{-2}\zeta t/u; p, \zeta)_\infty}{(q^2u/t; p, \zeta)_\infty (q^2\zeta t/u; p, \zeta)_\infty}, \quad (\text{B.1})$$

$$\mathcal{N}_S = \frac{(q^{-2}\zeta; p, \zeta)_\infty}{(q^2\zeta; p, \zeta)_\infty}, \quad (\text{B.2})$$

$$G_{S\Phi}(t, z) = (q^{-2}t)^{-1/k+2} \frac{(q^{k+1}z/t; p, \zeta)_\infty (q^{-k+1}\zeta t/z; p, \zeta)_\infty}{(q^{k-1}u/t; p, \zeta)_\infty (q^{-k-1}\zeta t/u; p, \zeta)_\infty}, \quad (\text{B.3})$$

$$G_{\Phi S}(z, t) = (q^{k-2}z)^{-1/k+2} \frac{(q^{-k+1}t/z; p, \zeta)_\infty (q^{k+1}\zeta z/t; p, \zeta)_\infty}{(q^{-k-1}t/z; p, \zeta)_\infty (q^{k-1}\zeta z/t; p, \zeta)_\infty}, \quad (\text{B.4})$$

$$G_\Phi(z, w) = \mathcal{N}_\Phi^2 (q^{k-2}z)^{1/2(k+2)} \frac{(q^4w/z; q^4, p, \zeta)_\infty (w/z; q^4, p, \zeta)_\infty}{(q^2w/z; q^4, p, \zeta)_\infty} \times \frac{(q^4\zeta z/w; q^4, p, \zeta)_\infty (\zeta z/w; q^4, p, \zeta)_\infty}{(q^2\zeta z/w; q^4, p, \zeta)_\infty}, \quad (\text{B.5})$$

$$\mathcal{N}_\Phi = \frac{(q^4\zeta; q^4, p, \zeta)_\infty (\zeta; q^4, p, \zeta)_\infty}{(q^2\zeta; q^4, p, \zeta)_\infty}, \quad (\text{B.6})$$

## C Another Derivation of Residue Formula

We here give another derivation of the residue formula (5.9) based on the integral formulae for the form factors obtained in Sec.4.

Let us consider the coefficients

$$\begin{aligned} & f_{\lambda\lambda_+\lambda\lambda_+\lambda}^{++--}(z_4, z_3, z_2, z_1) \\ &= \oint \frac{du_1}{2\pi i} \oint \frac{du_2}{2\pi i} < \Phi_{\lambda_+,+}^\lambda(z_4) \Phi_{\lambda_+,+}^{\lambda_+}(z_3) [\Phi_{\lambda_+,+}^\lambda(z_2), J^+(u_1)]_q [\Phi_{\lambda_+,+}^{\lambda_+}(z_1), J^+(u_2)]_q >_\lambda, \end{aligned} \quad (\text{C.1})$$

where we used the abridged notation

$$< \mathcal{O} >_\lambda = \text{Tr}_{\mathcal{H}_\lambda} (q^{4L_0} q^{-J_0^3} \mathcal{L}_\lambda \mathcal{O}). \quad (\text{C.2})$$

The integrand in (C.1) consists of the following four terms.

$$< \Phi_{\lambda_+,+}^\lambda(z_4) \Phi_{\lambda_+,+}^{\lambda_+}(z_3) \Phi_{\lambda_+,+}^\lambda(z_2) J^+(u_1) \Phi_{\lambda_+,+}^{\lambda_+}(z_1) J^+(u_2) >_\lambda, \quad (\text{C.3})$$

$$-q < \Phi_{\lambda_+,+}^\lambda(z_4) \Phi_{\lambda_+,+}^{\lambda_+}(z_3) J^+(u_1) \Phi_{\lambda_+,+}^\lambda(z_2) \Phi_{\lambda_+,+}^{\lambda_+}(z_1) J^+(u_2) >_\lambda, \quad (\text{C.4})$$

$$-q < \Phi_{\lambda_+,+}^\lambda(z_4) \Phi_{\lambda_+,+}^{\lambda_+}(z_3) \Phi_{\lambda_+,+}^\lambda(z_2) J^+(u_1) J^+(u_2) \Phi_{\lambda_+,+}^{\lambda_+}(z_1) >_\lambda, \quad (\text{C.5})$$

$$q^2 < \Phi_{\lambda_+,+}^\lambda(z_4) \Phi_{\lambda_+,+}^{\lambda_+}(z_3) J^+(u_1) \Phi_{\lambda_+,+}^\lambda(z_2) J^+(u_2) \Phi_{\lambda_+,+}^{\lambda_+}(z_1) >_\lambda. \quad (\text{C.6})$$



Let us investigate a pole structure at the point  $q^{-2}z_2/z_3 = 1$ . There are two sources of the poles. The first one is the correlation between the vertex  $\Phi_{1,0}(z)$  and the screening operator  $S(t)$ . The second one is the contour integral with respect to the argument of  $J^+(u)$ .

We first consider the poles arising from the product  $\Phi_{\lambda,+}^{\lambda+}(z_3)J^+(u_1)\Phi_{\lambda,+}^{\lambda-}(z_2)$  in (C.4) and (C.6). Let us consider the poles from the first source. Using (B.4),(4.19) and (3.14) for  $\Phi_{\lambda,+}^{\lambda-}(z_2)$ , the trace of the product  $\Phi_{1,0}(z_3)\Phi_{1,0}(z_2)S_{\delta}(t)$  contributes the factors:

$$G_{\Phi S}(z_3, t)G_{\Phi S}(z_2, t) \quad (C.7)$$

in the  $\Phi$  sector, and

$$\frac{E_q(Z_3, T; -1)E_q(Z_2, T; -1)}{E(Z_3, T^{\delta})E(Z_2, T^{\delta})} \quad (C.8)$$

in the  $\phi\chi$  sector. After scaling  $t \rightarrow q^{k-1}z_2t$ , we find that a simple pole at  $q^{-2}z_2/z_3 = 1, t = 1$  arising from  $G_{\Phi S}(z_3, t)$  cancelled by a zero at  $t = 1$  in  $G_{\Phi S}(z_2, t)$  as well as, in (C.8) with  $\delta = 1$ , a simple pole at  $t = 1$  cancelled by a zero at  $q^{-2}z_2/z_3 = 1, t = 1$ . As a result, we have no poles arising from the first source in (C.4) and (C.6).

Next let us consider the poles from the second source in (C.4) and (C.6). Let us first consider the integral with respect to  $u_1$ . The poles in the  $u_1$ -plane arise from the correlations among  $J^+(u_1)$ ,  $\Phi_{1,0}(z_2)$ ,  $\Phi_{1,0}(z_3)$  and  $S(t)$ . By using the formula (4.19), we find the trace of the product  $\Phi_{1,0}(z_3)J^+(u_1)\Phi_{1,0}(z_2)$  contributes the factor

$$\frac{E_q(Z_3, U; 1)E_q(U, Z_2; 1)}{E(Z_3, U^{\rho})E(U^{\rho}, Z_2)} = \begin{cases} \frac{(q^{k+5}z_3/u; q^4)_{\infty}(q^{k+1}z_2/u; q^4)_{\infty}}{(q^{k+3}z_3/u; q^4)_{\infty}(q^{k-1}z_2/u; q^4)_{\infty}} & \text{for } \rho = 1 \\ \frac{(q^{-k+1}u/z_3; q^4)_{\infty}(q^{-k+5}u/z_2; q^4)_{\infty}}{(q^{-k-1}u/z_3; q^4)_{\infty}(q^{-k+3}u/z_2; q^4)_{\infty}} & \text{for } \rho = -1 \end{cases} \quad (C.9)$$

whereas the trace of  $J^+(u_1)S(t)$  has a contribution by

$$\frac{E(U^{\rho}, T^{\delta})}{E(U, T)} = \begin{cases} 1 & \text{for } \delta = \rho = \pm 1 \\ q^{-1} \frac{(q^{k+1}z_2t/u; q^4)_{\infty}(q^{-k+3}u/z_2t; q^4)_{\infty}}{(q^{k-1}z_2t/u; q^4)_{\infty}(q^{-k+5}u/z_2t; q^4)_{\infty}} & \text{for } \delta = 1, \quad \rho = -1 \\ q \frac{(q^{k-3}z_2t/u; q^4)_{\infty}(q^{-k+7}u/z_2t; q^4)_{\infty}}{(q^{k-1}z_2t/u; q^4)_{\infty}(q^{-k+5}u/z_2t; q^4)_{\infty}} & \text{for } \delta = -1, \quad \rho = 1 \end{cases} \quad (C.10)$$

We hence find that, only in the case  $\delta = 1, \rho = -1$ , the contour is pinched at  $q^{-2}z_2/z_3 = 1, t = 1$  by the simple poles at  $u_1 = q^{k-1}z_2$  inside of the contour and the one at  $u_1 = q^{k+1}z_3$  outside of the contour. This implies the simple pole at  $q^{-2}z_2/z_3 = 1$  when  $t = 1$  in (C.4) and (C.6).

Next let us consider the contour integral with respect to  $u_2$ . For this purpose, we manipulate (C.4) to obtain

$$-q^2 \frac{1 - q^{k-1}z_1/u_2}{1 - q^{k+1}z_1/u_2} r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \sum_{\mu, \nu} W_{\mu}^{+}\left(\frac{z_4}{z_3}\right) W_{\nu}^{\mu}\left(\frac{z_1}{q^4 z_3}\right)$$

$$\times < \Phi_{\lambda,+}^{\mu}(z_4)J^+(u_1)\Phi_{\lambda,+}^{\lambda}(z_2)J^+(u_2)\Phi_{\nu,+}^{\lambda+}(q^4z_3)\Phi_{\mu,+}^{\nu}(z_1) >_{\mu}, \quad (\text{C.11})$$

where  $W_{\mu}^{+}(z) \equiv W \left( \begin{smallmatrix} \lambda & \lambda_+ \\ \mu & \lambda \end{smallmatrix} \middle| z \right)$  and  $W_{\nu}^{\mu}(z) \equiv W \left( \begin{smallmatrix} \mu & \lambda \\ \nu & \lambda_+ \end{smallmatrix} \middle| z \right)$ . Here we used the  $R$ -matrix symmetry (3.29), the commutation relation (4.1), the cyclic property of trace and the following relations.

$$J^+(u)\Phi_{1,0}(z) = q^{-1} \frac{1 - q^{k+1}z/u}{1 - q^{k-1}z/u} \Phi_{1,0}(z)J^+(u), \quad (\text{C.12})$$

$$[\Phi_{1,0}(z), J^+(u)]_q = -q^k z/u \frac{1 - q^2}{1 - q^{k+1}z/u} J^+(u)\Phi_{1,0}(z). \quad (\text{C.13})$$

For (C.6), the analogous manipulation leads to

$$\begin{aligned} & r \left( \frac{z_4}{z_3} \right) r \left( \frac{z_1}{q^4 z_3} \right) \sum_{\mu, \nu} W_{\mu}^{+} \left( \frac{z_4}{z_3} \right) W_{\nu}^{\mu} \left( \frac{z_1}{q^4 z_3} \right) \\ & \times < \Phi_{\lambda,+}^{\mu}(z_4)J^+(u_1)\Phi_{\lambda,+}^{\lambda}(z_2)J^+(u_2)\Phi_{\nu,+}^{\lambda+}(q^4z_3)\Phi_{\mu,+}^{\nu}(z_1) >_{\mu}, \end{aligned} \quad (\text{C.14})$$

Using (3.15) for  $\Phi_{\lambda,+}^{\lambda}(z_2)$ , the same analysis for the product  $\Phi_{\lambda,+}^{\lambda}(z_2)J^+(u_2)\Phi_{\nu,+}^{\lambda+}(q^4z_3)$  as the above implies that, only in the case  $\nu = \lambda$  with  $\delta = -1$ ,  $\rho = 1$ , pinching of the contour with respect to  $u_2$  occurs and a simple pole at  $q^{-2}z_2/z_3 = 1$  arises.

We now calculate the corresponding residues. Remarkably, the following operator identities are verified.

$$: \Phi_{1,0}(z_3)J_{-1}^+(u)\Phi_{1,0}(z_2)S_1(q^{k-1}z_2t) := 1 \quad (\text{C.15})$$

at  $t = 1$ ,  $q^{-2}z_2/z_3 = 1$  and  $u = q^{k-1}z_2$ , and

$$: \Phi_{1,0}(z_2)J_{-1}^+(u)\Phi_{1,0}(q^4z_3)S_1(q^{k+3}z_3t) := 1 \quad (\text{C.16})$$

at  $t = 1$ ,  $q^{-2}z_2/z_3 = 1$  and  $u = q^{k+3}z_3$ .

$$: S_{-1}(q^{k+1}z_3t)\Phi_{1,0}(z_3)J_1^+(u)\Phi_{1,0}(z_2) := 1 \quad (\text{C.17})$$

at  $t = 1$ ,  $q^{-2}z_2/z_3 = 1$  and  $u = q^{k-1}z_2$ , and

$$: S_{-1}(q^{k+1}z_2t)\Phi_{1,0}(z_2)J_1^+(u)\Phi_{1,0}(q^4z_3) := 1 \quad (\text{C.18})$$

at  $t = 1$ ,  $q^{-2}z_2/z_3 = 1$  and  $u = q^{k+3}z_3$ .

Due to these identities, we obtain the residues of the terms (C.4) and (C.6) as follows.

$$\begin{aligned} & \mathcal{N}_{\lambda,+}^{\lambda}(+, -) \oint \frac{du}{2\pi i} < \Phi_{\lambda,+}^{\lambda}(z_4)\Phi_{\lambda,+}^{\lambda+}(z_1)J^+(u) >_{\lambda} \\ & - q^2 \mathcal{N}_{\lambda,+}^{\lambda+}(-, +) \frac{1 - z_1/q^4z_3}{1 - q^2z_1/q^4z_3} r \left( \frac{z_4}{z_3} \right) r \left( \frac{z_1}{q^4z_3} \right) \sum_{\mu=\lambda_+, \lambda_-} W_{\mu}^{+} \left( \frac{z_4}{z_3} \right) W_{\lambda}^{\mu} \left( \frac{z_1}{q^4z_3} \right) \\ & \times \oint \frac{du}{2\pi i} < \Phi_{\lambda,+}^{\mu}(z_4)J^+(u)\Phi_{\mu,+}^{\lambda}(z_1) >_{\mu}, \end{aligned} \quad (\text{C.19})$$

for (C.4) and

$$\begin{aligned}
& \mathcal{N}_{\lambda_+}^{\lambda} (+, -) q^2 \oint \frac{du}{2\pi i} < \Phi_{\lambda_+,+}^{\lambda}(z_4) J^+(u) \Phi_{\lambda_+,+}^{\lambda_+}(z_1) >_{\lambda} \\
& + q^2 \mathcal{N}_{\lambda}^{\lambda_+} (-, +) r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \sum_{\mu=\lambda_+, \lambda_-} W_{\mu}^+ \left(\frac{z_4}{z_3}\right) W_{\lambda}^{\mu} \left(\frac{z_1}{q^4 z_3}\right) \\
& \times \oint \frac{du}{2\pi i} < \Phi_{\lambda_+,+}^{\mu}(z_4) J^+(u) \Phi_{\mu,+}^{\lambda}(z_1) >_{\mu}, \tag{C.20}
\end{aligned}$$

for (C.6).

For the remaining terms (C.3) and (C.5), we make analogous manipulation to (C.11) and (C.14):

$$\begin{aligned}
& q^{-2} \frac{1 - q^{k-1} z_1/u_2}{1 - q^{k+1} z_1/u_2} \frac{1 - q^{k+5} z_3/u_2}{1 - q^{k+3} z_3/u_2} r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \sum_{\mu, \nu} W_{\nu}^+ \left(\frac{z_4}{z_3}\right) W_{\lambda}^{\mu} \left(\frac{z_1}{q^4 z_3}\right) \\
& \times < \Phi_{\lambda_+,+}^{\mu}(z_4) \Phi_{\lambda_+,+}^{\lambda}(z_2) J^+(u_1) \Phi_{\nu,+}^{\lambda_+}(q^4 z_3) J^+(u_2) \Phi_{\mu,+}^{\nu}(z_1) >_{\mu} \tag{C.21}
\end{aligned}$$

and

$$\begin{aligned}
& q^2 \frac{1 - q^{k-1} z_1/u_2}{1 - q^{k+1} z_1/u_2} \frac{1 - q^{k-1} z_2/u_1}{1 - q^{k+1} z_2/u_1} r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \sum_{\mu, \nu} W_{\nu}^+ \left(\frac{z_4}{z_3}\right) W_{\nu}^{\mu} \left(\frac{z_1}{q^4 z_3}\right) \\
& \times < \Phi_{\lambda_+,+}^{\mu}(z_4) J^+(u_1) \Phi_{\lambda_+,+}^{\lambda}(z_2) J^+(u_2) \Phi_{\nu,+}^{\lambda_+}(q^4 z_3) \Phi_{\mu,+}^{\nu}(z_1) >_{\mu} \tag{C.22}
\end{aligned}$$

for (C.3), whereas

$$\begin{aligned}
& - \frac{1 - q^{k+5} z_3/u_2}{1 - q^{k+3} z_3/u_2} r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \sum_{\mu, \nu} W_{\mu}^+ \left(\frac{z_4}{z_3}\right) W_{\nu}^{\mu} \left(\frac{z_1}{q^4 z_3}\right) \\
& \times < \Phi_{\lambda_+,+}^{\mu}(z_4) \Phi_{\lambda_+,+}^{\lambda}(z_2) J^+(u_1) \Phi_{\nu,+}^{\lambda_+}(q^4 z_3) J^+(u_2) \Phi_{\mu,+}^{\nu}(z_1) >_{\mu} \tag{C.23}
\end{aligned}$$

and

$$\begin{aligned}
& - q^2 \frac{1 - q^{k-1} z_2/u_1}{1 - q^{k+1} z_2/u_1} r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \sum_{\mu, \nu} W_{\nu}^+ \left(\frac{z_4}{z_3}\right) W_{\mu}^+ \left(\frac{z_1}{q^4 z_3}\right) \\
& \times < \Phi_{\lambda_+,+}^{\mu}(z_4) J^+(u_1) \Phi_{\lambda_+,+}^{\lambda}(z_2) J^+(u_2) \Phi_{\nu,+}^{\lambda_+}(q^4 z_3) \Phi_{\mu,+}^{\nu}(z_1) >_{\mu} \tag{C.24}
\end{aligned}$$

for (C.5).

By the same analysis as before, we find only a simple pole at  $q^{-1} z_2/z_3 = 1$  in (C.21)~(C.24), and we obtain the following residues.

$$\begin{aligned}
& \mathcal{N}_{\lambda}^{\lambda_+} (-, +) \left\{ \frac{1 - q^{k+5} z_3/u}{1 - q^{k+3} z_3/u} \frac{1 - q^{k-1} z_1/u}{1 - q^{k+1} z_1/u} + q^2 \frac{1 - q^{k-1} z_2/u}{1 - q^{k+1} z_2/u} \frac{1 - z_1/q^4 z_3}{1 - q^2 z_1/q^4 z_3} \right\} r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \\
& \times \sum_{\mu=\lambda_+, \lambda_-} W_{\mu}^+ \left(\frac{z_4}{z_3}\right) W_{\lambda}^{\mu} \left(\frac{z_1}{q^4 z_3}\right) \oint \frac{du}{2\pi i} < \Phi_{\lambda_+,+}^{\mu}(z_4) J^+(u) \Phi_{\mu,+}^{\lambda}(z_1) >_{\mu} \tag{C.25}
\end{aligned}$$

from (C.3) and

$$\begin{aligned} \mathcal{N}_\lambda^{\lambda_+}(-, +) & \left\{ -\frac{1 - q^{k+5}z_3/u}{1 - q^{k+3}z_3/u} - q^2 \frac{1 - q^{k-1}z_2/u}{1 - q^{k+1}z_2/u} \right\} r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \\ & \times \sum_{\mu=\lambda_+, \lambda_-} W_\mu^+\left(\frac{z_4}{z_3}\right) W_\lambda^\mu\left(\frac{z_1}{q^4 z_3}\right) \oint \frac{du}{2\pi i} < \Phi_{\lambda,+}^\mu(z_4) J^+(u) \Phi_{\mu,+}^\lambda(z_1) >_\mu \end{aligned} \quad (\text{C.26})$$

from (C.5).

Combining the results (C.19)~(C.26), we finally obtain the residue

$$\begin{aligned} q^{-2}z_2/z_3 &= 1 \overset{\text{res}}{f}_{\lambda\lambda_+\lambda\lambda_+\lambda}(z_4, z_3, z_2, z_1) \\ &= \mathcal{N}_{\lambda_+}^\lambda(+, -) < \Phi_{\lambda_+,+}^\lambda(z_4) \Phi_{\lambda,-}^{\lambda_+}(z_1) >_\lambda \\ & \quad - q \mathcal{N}_\lambda^{\lambda_+}(-, +) \frac{1 - z_1/q^4 z_3}{1 - q^2 z_1/q^4 z_3} r\left(\frac{z_4}{z_3}\right) r\left(\frac{z_1}{q^4 z_3}\right) \sum_{\mu=\lambda_+, \lambda_-} W_\mu^+\left(\frac{z_4}{z_3}\right) W_\lambda^\mu\left(\frac{z_1}{q^4 z_3}\right) \\ & \quad \times < \Phi_{\lambda,+}^\mu(z_4) \Phi_{\mu,-}^\lambda(z_1) >_\mu \end{aligned} \quad (\text{C.27})$$

From (C.27), the residue at  $q^{-2}z_1/z_2 = 1$  is obtained as follows. Moving the vertex  $\Phi_{\lambda,-}^{\lambda_+}(z_1)$  and  $\Phi_{\lambda_+,+}^\lambda(z_1)$  in the both sides to the left by using the trace property and renaming the arguments as  $z_4 \rightarrow z_3, z_3 \rightarrow z_2, z_2 \rightarrow z_1$  and  $z_1 \rightarrow q^4 z_4$ , we obtain

$$\begin{aligned} q^{-2}z_1/z_2 &= 1 \overset{\text{res}}{f}_{\lambda_+\lambda\lambda_+\lambda\lambda_+}(z_4, z_3, z_2, z_1) \\ &= \mathcal{N}_{\lambda_+}^\lambda(+, -) f_{\lambda_+\lambda\lambda_+}^{+-}(z_4, z_3) \\ & \quad - \mathcal{N}_\lambda^{\lambda_+}(-, +) R_{-+}^{-+}\left(\frac{z_4}{z_2}\right) R_{++}^{++}\left(\frac{z_3}{z_2}\right) \sum_{\mu=\lambda_+, \lambda_-} W_\lambda^\mu\left(\frac{z_4}{z_2}\right) W_\mu^+\left(\frac{z_3}{z_2}\right) f_{\lambda\mu\lambda}^{-+}(z_4, z_3) \end{aligned} \quad (\text{C.28})$$

This is the special case of the general result (5.9). The general formula can be obtained from (C.28) by using the  $R$ -matrix symmetry.

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